SIMPLE RIESZ GROUPS OF RANK ONE HAVING WILD INTERVALS

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Abstract. We prove that every partially ordered simple group of rank one which is not Riesz embeds into a simple Riesz group of rank one if and only if it is not isomorphic to the additive group of the rationals. Using this result, we construct examples of simple Riesz groups of rank one G, containing unbounded intervals $(D_n)_{n\geq 1}$ and D, that satisfy: (a) For each $n \geq 1$, $tD_n \neq G^+$ for every $t < q_n$, but $q_nD_n = G^+$ (where (q_n) is a sequence of relatively prime integers); (b) For every $n \geq 1$, $nD \neq G^+$. We sketch some potential applications of these results in the context of K-Theory.

Introduction

One of the subjects of interest in the theory of partially ordered abelian groups is the analysis of intervals, that is, non-empty, upward directed and order-hereditary subsets. These have been used in instances of quite different flavour. For example, in [9] and [10], they proved to be essential in studying extensions of dimension groups. In the papers [11] (see also [6]), [19], [2] and [14], their usage was directed towards an understanding of the non-stable K-Theory of multipliers of C^*-algebras with real rank zero and von Neumann regular rings, basically by describing the monoid of equivalence classes of projections. Other applications can be found in [30], where the Riesz refinement property in monoids of intervals is studied in detail; in [29], where a complete description of the universal theory of Tarski's equidecomposability types semigroups is given, and also in [31], as an instrument to obtain some extensions of Edwards' Separation Theorem (see, e.g. [8, Theorem 11.13]).

Since, as just mentioned, these monoids appear useful in the context of K-Theory of operator algebras, there is a strong need for constructing explicit examples of such monoids that help in providing evidence towards the study of certain conditions in multiplier algebras. In this paper we present such examples, in the form of countable Riesz groups of rank one whose monoids of intervals enjoy certain relevant properties, thus adding new examples to the knowledge of Riesz groups. Our motivation for the search of these examples can also be traced back to the following question, asked by Goodearl in [8, Open Problem 30]: *Can every partially ordered simple abelian group be embedded in a simple Riesz group ?* This was proved

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to be the case by Wehrung ([32]) via a cofinal embedding. However, the fact that part of this construction depends on Model theoretical arguments prompts the need of finding more concrete realisations of these type of embeddings. More concretely, the embedding result just mentioned was used in [32, Example 3.14] to obtain an example of a torsion-free simple Riesz group G containing an interval $D \neq G^+$ such that $2D = G^+$. Wehrung then asked whether *such example can be realized as a torsion-free Riesz group of rank one (i.e. with positive cone isomorphic to an additive submonoid of* $\mathbb{Q}$*)*, see [32, Problem 3.15]. This question was answered by the second author in [17] by constructing a large family of simple groups that can be embedded into simple Riesz groups of rank one ([17, Theorem 2.11]).

We first extend this result, by showing that *an ordered simple group of rank one* (G, G^+) *– which is not Riesz – can be embedded into a simple Riesz group of rank one if and only if* $G \not\cong \mathbb{Q}$. This is done in Section 1, and used subsequently to provide wide generalizations of Wehrung's example. The main tool in [32, Example 3.14] is the construction of a certain submonoid of $\mathbb{Q}^+$ using the submonoid of $\mathbb{Z}^+$ generated by 2 and 7. We extend this construction in Section 2 to combinations of submonoids of $\mathbb{Z}^+$ generated by coprime integers p and q, but with considerable more extra care needed. This provides us with an example of a simple ordered group (G, G^+) that contains a proper interval D, a multiple of which equals the positive cone G^+. However, this is not a Riesz group. An inductive procedure, based on taking direct limits of this type of construction, leads in Section 3 to a first example of a Riesz group for which a whole sequence of (proper and unbounded) intervals (D_n) can be constructed; every such interval has the property that $tD_n \neq G^+$ for every $t \leq q_n$, and $q_nD_n = G^+$. Here, (q_n) is an increasing sequence of relatively prime integers. The inductive step is based on a suitable amalgamation of groups (of the type considered in Section 2, which is why we can refer to them as basic building blocks) in a commutative diagram. A further modification of this example, after using the refinement property on the monoid of countably generated intervals, allows us to achieve that the sequence of intervals is moreover decreasing.

In Section 4 we state some arithmetic results on simple components (see below), which are used in Section 5 in order to construct an example of a simple Riesz group G containing an unbounded interval $D \subset G^+$ such that $nD \neq G^+$ for every $n \geq 1$. The constructions carried out in Sections 3 and 5 are combined in Section 6 to obtain an example of a simple Riesz group that has the properties exhibited in Sections 3 and 5.

An object which is central in this paper is that of a *simple component*, as our examples are built essentially via direct limit constructions of simple components of various kinds. In short, a simple component is nothing else but the group $\mathbb{Z}$, together with a partial ordering that makes it simple. This, for example, includes $(\mathbb{Z}, \langle k, l\rangle)$, where $\langle k, l\rangle$ is the submonoid of $\mathbb{Z}$ generated by two relatively prime integers. Simple components have been studied in different contexts, notably with relation to K-theoretical aspects of C^*-algebras (see, e.g. [26], [24], [7], where it is shown that there exist simple C^*-algebras with stable rank one whose K_0 groups are simple components). Consider also the following question:

Question 1. ([25]) Let $N \in \mathbb{N}$. For every $1 \leq i \leq N$ take q_i and m_i in $\mathbb{N}$ to be relatively prime, where q_i is prime. Take moreover a positive integer L that is coprime with each q_i and m_i. Consider the following subsemigroup of the positive integers:

$$S = \frac{1}{L}\left(\bigcap_{i=1}^{N}\langle q_i, m_i\rangle\right) \cap \mathbb{Z}\,.$$

Can every positive cone of a simple component be expressed as S for suitable choices of N, (q_i), (m_i) and L ?

The construction technique developed by Toms in [25] produces, for every such S as above, a simple C^*-algebra with stable rank one whose K_0 group is isomorphic to $\mathbb{Z}$ with positive cone S. The real rank of these examples is not known, but it seems safe to predict it is not zero (this is more so taking into account the recent examples by Rørdam, see [22], [23]). Hence, these results suggest the problem of constructing simple C^*-algebras A with real rank zero and stable rank one such that $(K_0(A), K_0(A)^+)$ is isomorphic to one of the groups we construct in this paper (as well as those constructed in [16] and [17]), by lifting connecting maps in the direct limit expression of these groups (as limits of simple components and order-embeddings), to C^*-algebra maps between C^*-algebras of the type constructed in [25]. Other relevant aspects of this discussion are outlined in Section 7.

Throughout the paper we will refer to [8] for notations and definitions on partially ordered abelian groups. We recall here some basic notions that we shall use frequently. A *cone* of an abelian group G is an additive submonoid P of G containing zero. We say that the cone P is *strict* if $P \cap (-P) = \{0\}$. A *partially ordered abelian group* is an abelian group G endowed with a strict cone G^+, called the *positive cone* of G. The usual notation for a partially ordered group is (G, G^+), and the elements of G^+ are referred to as the *positive elements* of G. The order induced by G^+ is denoted in this paper by $\leq_G$. We say that (G, G^+) is *directed* provided that any element can be written as a difference of two positive elements. An element u in G is said to be an *order-unit* provided that $0 \neq u \in G^+$ and for each element x in G, there exists n in $\mathbb{N}$ such that $-nu \leq_G x \leq_G nu$ (note that G will then be directed). A partially ordered abelian group is said to be *simple* when it is non-zero and every non-zero positive element is an order-unit. A partially ordered abelian group (G, G^+) satisfies the *Riesz decomposition property* (or is a *Riesz group*, for short) if the following condition is met in G^+: whenever $x \leq_G y_1 + y_2$ in G^+, there exist x_1 and x_2 in G^+ such that $x = x_1 + x_2$ and $x_j \leq_G y_j$ for all j. It is well-known that this is equivalent to the Riesz refinement and interpolation properties (see, e.g [8, Proposition 2.1]).

If (G, G^+) and (H, H^+) are partially ordered abelian groups, a *positive morphism* is a group homomorphism $f\colon G \to H$ such that $f(G^+) \subseteq H^+$. A positive morphism $f\colon G \to H$ is an *order-embedding* if f is one-to-one and $x \in G^+$ whenever $f(x) \in H^+$ (in other words, $f(G^+) = f(G) \cap H^+$).

1. Embedding results

In this section, we will establish some results about embedding simple groups into simple Riesz groups, that improve those appearing in [17]. The first one was shown by the second author in a rather complicated way [unpublished]. The proof we present here was pointed out by G. Bergman.

We start by recalling some basic facts related to generalized integers (see, e.g. [16]). Let $\mathbb{P}$ be the set of the natural prime numbers. A *generalized integer* $\mathfrak{n}$ is a map

$$\mathfrak{n}\colon \mathbb{P} \to \{0, 1, 2, \ldots, \infty\}\,.$$

Usually we write

$$\mathfrak{n} = \prod_{p \in \mathbb{P}} p^{\mathfrak{n}(p)}. \tag{1.1}$$

When $\mathfrak{n}$ is finite (i.e. it never takes the value ∞ and it is zero except at finitely many primes), we identify $\mathfrak{n}$ with the integer appearing on the right hand side of (1.1). Multiplication extends naturally to generalized integers, namely, the product $\mathfrak{n} \cdot \mathfrak{m}$ of $\mathfrak{n}$ and $\mathfrak{m}$ is defined as $(\mathfrak{n} \cdot \mathfrak{m})(p) = \mathfrak{n}(p) + \mathfrak{m}(p)$ for every p in $\mathbb{P}$. Thus we say that $\mathfrak{n}$ divides $\mathfrak{m}$, in symbols $\mathfrak{n} \mid \mathfrak{m}$, if there is $\mathfrak{n}'$ such that $\mathfrak{m} = \mathfrak{n} \cdot \mathfrak{n}'$. We say that $\mathfrak{n}$ and $\mathfrak{m}$ are coprime if for every p in $\mathbb{P}$ we have $0 \in \{\mathfrak{n}(p), \mathfrak{m}(p)\}$.

Given a generalized integer $\mathfrak{n}$, we associate to it an additive subgroup of $\mathbb{Q}$ containing 1 by setting $\mathbb{Z}_{\mathfrak{n}} = \{\frac{a}{b} \in \mathbb{Q} \mid a \in \mathbb{Z} \text{ and } b \mid \mathfrak{n}\}$. Conversely, one can associate a generalized integer to any subgroup of $\mathbb{Q}$ that contains 1, and these assignements are mutually inverse (see [16, Lemma 2.3]).

Given a sequence $A = (a_n)_{n\geq 1}$, we define $\mathfrak{n}(A) = \prod_{n\geq 1} a_n$, and we say that the sequence A is the sequence associated to $\mathfrak{n}(A)$. A sequence $A = (a_n)$ is associated to a generalized integer $\mathfrak{n}$ when $\mathfrak{n} = \mathfrak{n}(A)$. We can always associate a sequence to a generalized integer, as shown in [16, Lemma 3.10].

One notion that will become relevant in this paper is that of a *simple component* ([16]). This is, by definition, the group $\mathbb{Z}$ with a positive cone G^+ such that $G = (\mathbb{Z}, G^+)$ is partially ordered and simple. It was proved in [24, Proposition 2.4 (ii)] and [24, Proposition 2.5] that, if $(\mathbb{Z}, G^+)$ is a simple component, then G^+ is the submonoid of $\mathbb{Z}^+$ generated by a (unique and minimal) finite set of elements $n_1, \ldots, n_k$ in $\mathbb{Z}^+$ (so that $G^+ = \langle n_1, \ldots, n_k\rangle$, and in fact $\gcd(n_1, \ldots, n_k) = 1$). In the particular case that $G^+ = \langle k, l\rangle$ (and thus k and l are coprime integers), then one can determine the smallest non-negative integer N for which $N + p \in G^+$ for all $p \geq 0$, but $N - 1 \notin G^+$ (see [24, Lemma 2.3]); namely, $N = kl - k - l$.

Proposition 1.1. *Every simple ordered group of rank one (G, G^+) is isomorphic (as an ordered group) to a direct limit of a directed system $(G_n, f_{n,n+1})$, such that $G_n = (\mathbb{Z}, G_n^+)$ is a simple component for every n in $\mathbb{N}$ and $f_{n,n+1}\colon G_n \to G_{n+1}$ is an order-embedding.*

Proof. Since G is an abelian group of rank one, we can assume without loss of generality that $1 \in G$. Thus, by [16, Lemma 2.4], there exists a (unique) generalized integer $\mathfrak{n} = \prod_{k\geq 1} a_k$ such that $G \cong \mathbb{Z}_{\mathfrak{n}}$. For each $n \geq 1$, let $b_n = \prod_{k=1}^{n} a_k$, and define $H_n = (1/b_n)\mathbb{Z}$. Notice that $H_n \subset H_{n+1}$ for each $n \geq 1$, and also that $G = \bigcup_{n\geq 1} H_n$. Now, for every $n \geq 1$, let $g_{n,n+1}\colon H_n \to H_{n+1}$ denote the natural inclusion map, and define $H_n^+ = G^+ \cap H_n$.

We claim that (H_n, H_n^+) is a simple group for each $n \geq 1$. To check this, pick a non-zero element x in H_n^+, and let $y \in H_n$ be any element. Since (G, G^+) is a simple group, there exists m in $\mathbb{N}$ such that $-mx \leq_G y \leq_G mx$. Thus, $mx - y, y + mx \in G^+ \cap H_n = H_n^+$, so that the previous inequality also holds in H_n, as desired.

We claim now that the map $g_{n,n+1}\colon H_n \to H_{n+1}$ is an order-embedding for every $n \geq 1$. By definition, it is a positive one-to-one map. Now, let $x \in H_n$ be an element such that $g_{n,n+1}(x) \in H_{n+1}^+ = G^+ \cap H_{n+1}$. Since $x = g_{n,n+1}(x) \in G^+$ and $x \in H_n$, we conclude that $x \in H_n^+$.

Finally, for each $n \geq 1$, let $f_n\colon \mathbb{Z} \to H_n$, given by multiplication by $(1/b_n)$. Define $G_n^+ = b_n H_n^+ \subseteq \mathbb{Z}$, and $G_n = (\mathbb{Z}, G_n^+)$. Then, $f_n\colon G_n \to H_n$ is an order-isomorphism, and hence the group G_n is a simple component. Moreover, for each $n \geq 1$, the map

$$f_{n,n+1} = f_{n+1}^{-1} \circ g_{n,n+1} \circ f_n \colon G_n \to G_{n+1}$$

is an order-embedding. Hence, for each $n \geq 1$ we get a commutative diagram

$$\begin{tikzcd} G_n \arrow[r, "f_{n,n+1}"] \arrow[d, "f_n"'] & G_{n+1} \arrow[d, "f_{n+1}"] \\ H_n \arrow[r, "g_{n,n+1}"] & H_{n+1} \end{tikzcd}$$

whence the maps f_n induce an order-isomorphism f from $\varinjlim((\mathbb{Z}, G_n^+), f_{n,n+1})$ onto (G, G^+), as wanted. $\square$

As mentioned in the Introduction, one of the main objectives in [17] was to study the embedding of a certain class of simple partially ordered groups of rank one into simple Riesz groups of rank one. Such groups are parametrized by a triple $(A, B, \mathcal{H})$, where $\mathcal{H}$ is a sequence of simple groups (basically $\mathbb{Z}$ with different positive cones) and A and B sequences of positive integers, all subject to certain axioms. The proof of the key embedding result, established in [17, Theorem 2.11], is based on the fact that these groups are isomorphic to a direct limit of an inductive system $((\mathbb{Z}, G_n^+), f_{n,n+1})$ such that, for every n in $\mathbb{N}$, the map $f_{n,n+1} \colon (\mathbb{Z}, G_n^+) \to (\mathbb{Z}, G_{n+1}^+)$ is an order-embedding given by multiplication by a non-negative integer a_n (where $A = (a_n)_{n\geq 1}$). Thus, in view of Proposition 1.1, we can strengthen [17, Theorem 2.11] as follows:

Theorem 1.2. *Let (G, G^+) be a simple ordered group of rank one, and let $\mathfrak{n}$ be the generalized integer associated to G. Given any infinite generalized integer $\mathfrak{m}$ coprime with $\mathfrak{n}$, there exist a simple Riesz group of rank one $(\widetilde{G}(\mathfrak{m}), \widetilde{G}^+(\mathfrak{m}))$ and a positive morphism*

$$\tau \colon G \to \widetilde{G}(\mathfrak{m})$$

such that:

(i) *The group $\widetilde{G}(\mathfrak{m})$ is isomorphic to $\mathbb{Z}_{\mathfrak{n}\cdot\mathfrak{m}}$ (as abelian groups).*
(ii) *The map τ is an order-embedding.* $\square$

The next result was also pointed out by G. Bergman.

Lemma 1.3. *Let $G_1 = (\mathbb{Q}, G_1^+)$ and $G_2 = (\mathbb{Q}, G_2^+)$ be partially ordered abelian groups, and let $f \colon G_1 \to G_2$ be a positive map. Then f is an order-embedding if and only if it is an isomorphism of ordered groups.*

Proof. Clearly, since f is a group morphism from $\mathbb{Q}$ to $\mathbb{Q}$, it is identically zero or an isomorphism of abelian groups.

Suppose that f is an order-embedding, so that in particular it is one-to-one. Hence the previous observation implies that f is an isomorphism. But then we also have

$$f(G_1^+) = G_2^+ \cap f(G_1) = G_2^+ \cap \mathbb{Q} = G_2^+ ,$$

so that it is an order-isomorphism. The converse is obvious. $\square$

A first consequence of Theorem 1.2 and Lemma 1.3 is the following characterization of embeddability of simple ordered groups into simple Riesz groups. This will be an important result in the sequel.

Theorem 1.4. *An ordered simple group of rank one (G,G^+) which is not a Riesz group can be embedded into a simple Riesz group of rank one if and only if $G \ncong \mathbb{Q}$.*

Proof. First, assume that (G,G^+) is a simple ordered group of rank one which is not a Riesz group, and suppose that $G \cong \mathbb{Q}$. Assume that (H,H^+) is a simple Riesz group of rank one and that $f\colon G \to H$ is an order-embedding. Then (H,H^+) is order-isomorphic to a subgroup of $(\mathbb{Q},\mathbb{Q}^+)$ and the composition of the isomorphism $\mathbb{Q} \cong G$ with f and the embedding of H into $\mathbb{Q}$ provides a non-zero morphism from $\mathbb{Q}$ to $\mathbb{Q}$. Evidently this must be an isomorphism, which implies that f is surjective. But then

$$f(G^+) = f(G) \cap H^+ = H \cap H^+ = H^+ ,$$

that is, f is an order-isomorphism. This implies that G is a Riesz group, a contradiction.

Conversely, suppose that $G \ncong \mathbb{Q}$. Since the generalized integer associated to $\mathbb{Q}$ is $\mathfrak{n}(\mathbb{Q}) = \prod_{p_i\in\mathbb{P}} p_i^{\infty}$, where $\mathbb{P}$ is the set of all non-negative prime numbers, we conclude that for the generalized integer associated to G, say $\mathfrak{n}(G) = \prod_{p_i\in\mathbb{P}} p_i^{n_i}$, there exists at least one prime number p_k so that $n_k < \infty$. Now, multiplication by $p_k^{n_k}$ defines an order-isomorphism from (G,G^+) onto $(p_k^{n_k}G, p_k^{n_k}G^+)$. Notice that $\mathfrak{n}(p_k^{n_k}G) = \mathfrak{n}(G)/p_k^{n_k}$, so that $\mathfrak{n}(p_k^{n_k}G)$ is coprime with p_k. Hence, applying Theorem 1.2 we get an order-embedding from $(p_k^{n_k}G, p_k^{n_k}G^+)$ into a simple Riesz group of rank one (H,H^+). Thus, the composition of both maps gives us an order-embedding from (G,G^+) into (H,H^+), as desired. □

2. Intervals in basic building blocks

This section, technical in nature, aims at the study of certain simple groups of rank one. These will be used as our basic building blocks in the subsequent sections, by connecting them through order-embeddings and forming various inductive limits. We shall focus on the construction of proper intervals in these groups such that a certain multiple (that can be controlled) equals the positive cone.

Let G be a partially ordered abelian group with positive cone G^+. A non-empty subset X of G^+ is called an *interval* in G^+ if X is upward directed and order-hereditary. We denote by $\Lambda(G^+)$ the set of intervals in G^+. Note that $\Lambda(G^+)$ becomes an abelian monoid with operation defined by $X + Y = \{z \in G^+ \mid z \leq x + y \text{ for some } x \text{ in } X, y \text{ in } Y\}$. An interval X in G^+ is said to be *generating* if every element of G^+ is a sum of elements from X. We say that X in $\Lambda(G^+)$ is *countably generated* provided that X has a countable cofinal subset (i.e. there is a sequence (x_n) of elements in X such that for any x in X there exists n in $\mathbb{N}$ with $x \leq_G x_n$). Notice that, since any interval is upward directed, if (x_n) is a countable cofinal subset generating an interval X, then we can choose a countable cofinal subset (y_n) generating X with the property that $y_n \leq_G y_{n+1}$ for all $n \geq 1$. We shall in this case use the notation $X = \langle y_n\rangle$. We denote by $\Lambda_\sigma(G^+)$ the set of all countably generated intervals in G^+.

Definition 2.1. Let p and q be positive integers such that $1 < q < p-q$ and that $\gcd(q,p) = 1$. Denote by $A = \langle q, p-q\rangle$ the submonoid of $\mathbb{Z}^+$ generated by q and $p-q$. Let $s \in A\setminus\{0\}$ and take r in $\mathbb{Z}^+$ be such that $1 < r < s-r$ and $\gcd(r,s) = 1$. Denote by $B = \langle r, s-r\rangle$.

Next, define M to be the submonoid of $\mathbb{Q}^+$ whose generators are fractions of the form

$$\frac{k}{r}\,,\ \text{where } k \in A\,,\ \text{and } \frac{k'}{r}\left(\frac{s}{r}\right)^n\,,\ \text{where } k' \in B \text{ and } n \geq 1\,.$$

Define (G, G^+) to be the group $G = M + (-M)$ with positive cone $G^+ = M$. Since G is also directed, it follows that G is a simple partially ordered abelian group. Notice that (G, G^+) is not a Riesz group.

For all n in $\mathbb{N}$, denote $e_n = (\frac{s}{r})^n$.

Lemma 2.2. *The set $D = \{x \in G^+ \mid x \leq_G e_n$ for some $n\}$ is a proper interval in G^+ such that $rD = G^+$.*

Proof. We first show that the sequence (e_n) is increasing. By construction, $e_n \in M$ for all n. Also, if $n \geq 1$, we have

$$e_{n+1} - e_n = \left(\frac{s}{r}\right)^n \left(\frac{s}{r} - 1\right) = \left(\frac{s}{r}\right)^n \left(\frac{s-r}{r}\right),$$

which is an element of M since $s - r \in B$. This proves that D is an interval in G^+.

We now prove that $s \notin D$, while it is clear that $s = \frac{sr}{r} \in M$. This will entail that D is proper. In order to achieve this, we proceed by induction. We evidently have that $e_1 - s = \frac{s(1-r)}{r} \notin M$ (because $1 - r < 0$). Assume, by way of contradiction, that $s \not\leq_G e_{m-1}$ for some $m \geq 2$, and that $s \leq_G e_m$. This means that we can find a natural number n, elements k_l in B for $l = 1, \dots, n$, and an element k in A such that

$$\left(\frac{s}{r}\right)^m - s = e_m - s = \sum_{l=1}^{n} \frac{k_l}{r} \left(\frac{s}{r}\right)^l + \frac{k}{r}. \tag{2.1}$$

We can obviously choose n above so that $k_n \neq 0$. Since $k_n \in B$, we obtain that $k_n \geq r$. Therefore, substituting k_n by r in (2.1) we get the following bound:

$$\left(\frac{s}{r}\right)^m > \left(\frac{s}{r}\right)^m - s \geq \left(\frac{s}{r}\right)^n.$$

This implies that $n < m$.

Now, the right-hand side of (2.1) belongs to $r^{-(n+1)}\mathbb{Z}^+$. Hence, after multiplying by r^{n+1} we get that

$$s^m r^{n+1-m} - s r^{n+1} = r^{n+1} \left(\left(\frac{s}{r}\right)^m - s\right) \in \mathbb{Z}^+.$$

Since $\gcd(r, s) = 1$, the above implies that $m \leq n + 1$. Thus $m = n + 1$.

We now claim that $r \nmid k_n$ and that $k_n < s$. Assume first that $r \mid k_n$. Then the right hand side of (2.1) would belong to $r^{-n}\mathbb{Z}^+$. Hence

$$\frac{s^{n+1}}{r} - r^n s = r^n \left(\left(\frac{s}{r}\right)^{n+1} - s\right) \in \mathbb{Z}^+,$$

contradicting the fact that r and s have no common factors.

Also, if $k_n \geq s$, then substituting k_n by s in (2.1) we get the following bound

$$\left(\frac{s}{r}\right)^{n+1} - s \geq \left(\frac{s}{r}\right)^{n+1},$$

which is impossible. The claim is therefore established.

¿From our claim and the fact that $B = \langle r, s - r \rangle$, it follows that $k_n = s - r$. Indeed, if we write $k_n = ar + b(s - r)$ for some positive integers a and b, we know that $b \neq 0$ since $r \nmid k_n$. If $a \neq 0$, then $s > k_n \geq r + s - r = s$, which is impossible. Hence $a = 0$. If now $b \geq 2$, then $s > k_n \geq 2(s - r)$, so $s - 2r < 0$, in contradiction of our election of r and s.

Finally

$$\begin{aligned}\left(\frac{s}{r}\right)^{n+1} - s &= \sum_{l=1}^{n} \frac{k_l}{r}\left(\frac{s}{r}\right)^{l} + \frac{k}{r}\\ &= \sum_{l=1}^{n-1} \frac{k_l}{r}\left(\frac{s}{r}\right)^{l} + \frac{s-r}{r}\left(\frac{s}{r}\right)^{n} + \frac{k}{r}\\ &= \sum_{l=1}^{n-1} \frac{k_l}{r}\left(\frac{s}{r}\right)^{l} \left(\frac{s}{r}\right)^{n+1} - \left(\frac{s}{r}\right)^{n} + \frac{k}{r}\,.\end{aligned}$$

This implies that

$$e_n - s = \sum_{l=1}^{n-1} \frac{k_l}{r}\left(\frac{s}{r}\right)^{l} + \frac{k}{r} \in M\,,$$

which contradicts our inductive hypothesis since $n = m-1$. Therefore, by induction, $s \not\leq_G e_m$ for all m and so $D \neq G^+$.

Next, we prove that $rD = M$. First, we claim that $2^k e_n \leq_G re_{n+k-1}$ for all n and all k. Indeed, if $k = 1$, then $2e_n \leq_G re_n$ for all n (since $r \geq 2$). Now assume that, for some $k \geq 2$, we have $2^k e_n \leq_G re_{n+k-1}$ for all n. Then

$$e_{n+k-1}\frac{r(s-2r)}{r} = r\left(\frac{s}{r}\right)^{n+k-1}\left(\frac{s}{r} - 2\right) = re_{n+k} - 2re_{n+k-1} \leq_G re_{n+k} - 2^{k+1}e_n\,.$$

Notice that by our choosing of r and s, we have $s - 2r = s - r - r > 0$. Therefore the element $e_{n+k-1}\frac{r(s-2r)}{r}$ belongs to M, and hence $re_{n+k} - 2^{k+1}e_n \in G^+$. By induction, the claim is proved.

Now take e_1, which belongs to D and is non-zero. Since G is simple, e_1 is an order-unit. Given x in G^+, there is then a natural number n such that $x \leq_G ne_1$. Choose k such that $n < 2^k$. Hence, using the previous claim we conclude that $x \leq_G ne_1 \leq_G 2^k e_1 \leq_G re_k$. This shows that $G^+ \subseteq rD$. Since the inclusion $rD \subseteq G^+$ is obvious, we get equality. □

Proposition 2.3. *Let D be the interval defined in Lemma 2.2. Then, for any $t \leq r-1$, we have $tD \neq G^+$, and $rD = G^+$.*

Proof. We have already checked in Lemma 2.2 that $rD = G^+$. We proceed by induction on t to prove that $s \notin tD$ for any $1 \leq t \leq r-1$. The case $t = 1$ is taken care of by the proof of Lemma 2.2. Next, assume that, if $i < t$, we have $s \not\leq_G ie_m$ for all m. We will prove that $s \not\leq_G te_m$ for all m, using induction on m.

Since $te_1 - s = t\frac{s}{r} - s = \frac{s(t-r)}{r} \notin M$, we see that $s \not\leq_G te_1$.

Now, assume that $m \geq 2$ and that $s \not\leq_G te_j$ for all $j < m$. By way of contradiction, assume that $te_m - s \in M$. This means that we can find a natural number n, and elements k_l in B for $l = 1, \ldots, n$, k in A, such that

$$t\left(\frac{s}{r}\right)^{m} - s = \sum_{l=1}^{n} \frac{k_l}{r}\left(\frac{s}{r}\right)^{l} + \frac{k}{r}\,. \tag{2.2}$$

Since $k_n \in B$, we have that $k_n \geq r$. The right-hand side of (2.2) belongs to $r^{-(n+1)}\mathbb{Z}^+$. Hence

$$tr^{n+1-m}s^m - r^{n+1}s = r^{n+1}\left(t\left(\frac{s}{r}\right)^{m} - s\right) \in \mathbb{Z}^+\,.$$

This, coupled with the assumptions that $t < r$ and $\gcd(r, s) = 1$, implies that $m \leq n+1$. We first deal with the case $m = n+1$. From (2.2), we get

$$\left(\frac{s}{r}\right)^n \left(\frac{ts - k_n}{r}\right) = \left(\frac{s}{r}\right)^n \left(t\,\frac{s}{r} - \frac{k_n}{r}\right) = \sum_{l=1}^{n-1} \frac{k_l}{r} \left(\frac{s}{r}\right)^l + \frac{k}{r} + s\,. \tag{2.3}$$

Since the right-hand side of the above belongs to $r^{-n}\mathbb{Z}^+$ we have that $s^n\left(\frac{ts-k_n}{r}\right) \in \mathbb{Z}^+$. As $r \nmid s$, we conclude that $r \mid ts - k_n$. Write $ts - k_n = t'r$ for some t' in $\mathbb{Z}^+$. Now we have $t'r + k_n - tr - t(s-r) = 0$. Adding $(s-r)r - (s-r) - r$ to this equality, we get

$$\begin{aligned} (s-r)r - (s-r) - r &= t'r + k_n - tr - t(s-r) + (s-r)r - (s-r) - r \\ &= r(t' - t - 1) + k_n + (s-r)(r-1-t)\,. \end{aligned} \tag{2.4}$$

By [24, Lemma 2.3] applied to the submonoid B, $(s-r)r - (s-r) - r \notin B$. On the other hand, $r - 1 - t \geq 0$ and so $(s-r)(r-1-t) \in B$. Since r, $k_n \in B$, it follows from (2.4) that $t' - t - 1 < 0$, that is, $t' < t+1$.

We now substitute $ts - k_n = t'r$ in (2.3). We obtain

$$\left(\frac{s}{r}\right)^n t' = \left(\frac{s}{r}\right)^n \left(\frac{t'r}{r}\right) = \sum_{l=1}^{n-1} \frac{k_l}{r} \left(\frac{s}{r}\right)^l + \frac{k}{r} + s\,,$$

whence $t'e_n - s = t'\left(\frac{s}{r}\right)^n - s \in M$, an absurdity since $n = m-1$ and $t' < t$.

Next we deal with the case $n = m + a$ where $a \geq 0$. We rewrite (2.2) as

$$t\left(\frac{s}{r}\right)^m - s = \sum_{l=1}^{m-1} \frac{k_l}{r} \left(\frac{s}{r}\right)^l + \sum_{l=m}^{m+a} \frac{k_l}{r} \left(\frac{s}{r}\right)^l + \frac{k}{r}\,,$$

that is,

$$\begin{aligned} &\sum_{l=1}^{m-1} \frac{k_l}{r} \left(\frac{s}{r}\right)^l + \frac{k}{r} + s \\ &= \left(\frac{s}{r}\right)^m \left(t - \sum_{l=m}^{m+a} \frac{k_l}{r} \left(\frac{s}{r}\right)^{l-m}\right) \\ &= \left(\frac{s}{r}\right)^m \left(\frac{tr^{a+1} - \sum_{l=m}^{m+a} k_l s^{l-m} r^{a+1-l+m}}{r^{a+1}}\right). \end{aligned} \tag{2.5}$$

Let $u = \sum_{l=m}^{m+a} k_l s^{l-m} r^{a+1-l+m}$. Since the left-hand side in (2.5) belongs to $r^{-m}\mathbb{Z}^+$, we obtain (after multiplying the right-hand side of the equality by r^m) that $r^{a+1} \mid tr^{a+1} - u$. Write $tr^{a+1} - u = t''r^{a+1}$, for t'' in $\mathbb{Z}^+$, and rearrange as $r^{a+1}(t'' - t) + u = 0$. Since $k_n = k_{m+a} \neq 0$, we have that $u > 0$. Therefore $t'' < t$. Finally, we substitute $tr^{a+1} - u = t''r^{a+1}$ in (2.5) and obtain

$$\sum_{l=1}^{m-1} \frac{k_l}{r} \left(\frac{s}{r}\right)^l + \frac{k}{r} = \left(\frac{s}{r}\right)^m \frac{t''r^{a+1}}{r^{a+1}} - s = t''e_m - s\,,$$

so that $s \leq_G t''e_m$ and $t'' < t$, a contradiction. $\square$

Proposition 1.1 allows us to write the group (G, G^+) as an inductive limit of simple components and order-embeddings. Below we present this representation in a way more related to the construction and that will be used in the next section.

Proposition 2.4. *The group (G, G^+) can be realized as a direct limit $\varinjlim((\mathbb{Z}, G_i^+), f_{i,i+1})$, where $(\mathbb{Z}, G_i^+)$ are simple components and the maps $f_{i,i+1}\colon G_i \to G_{i+1}$ are order-embeddings given by multiplication by r.*

Proof. Let $G_0^+ = A$ and set $G_i^+ = rG_{i-1}^+ + s^i B$ if $i \geq 1$. Since $\gcd(r, s) = 1$, the groups $(\mathbb{Z}, G_i^+)$ are simple components for all i, and the maps $f_{i,i+1}$ given by multiplication by r are order-embeddings ([17, Lemma 2.3]).

Next, define $H_0 = \frac{1}{r}\mathbb{Z}$, $H_0^+ = \frac{1}{r}A$, and $H_i = (\frac{1}{r})^i\mathbb{Z}$ and $H_i^+ = H_{i-1}^+ + \frac{1}{r}(\frac{s}{r})^i B$ if $i \geq 1$.

Clearly, we have the following commutative diagram:

$$\begin{array}{ccccccc}
(H_0, H_0^+) & \longrightarrow & (H_1, H_1^+) & \longrightarrow & (H_2, H_2^+) & \longrightarrow & \cdots \\
{\scriptstyle r\cdot}\downarrow & & \downarrow{\scriptstyle r^2\cdot} & & \downarrow{\scriptstyle r^3\cdot} & & \\
(\mathbb{Z}, G_0^+) & \xrightarrow{f_{0,1}} & (\mathbb{Z}, G_1^+) & \xrightarrow{f_{1,2}} & (\mathbb{Z}, G_2^+) & \xrightarrow{f_{2,3}} & \cdots
\end{array}$$

where the maps in the top row are given by inclusions and all columns are order-isomorphisms. The limit of the top row is (G, G^+) and it follows that the natural induced map to the limit of the bottom row is an order-isomorphism, as desired. $\square$

Remark 2.5. Following [17, Section 2], it is easy to see that (G, G^+) is order-isomorphic to the group $(G(A', B', \mathcal{H}'), G^+(A', B', \mathcal{H}'))$ associated to the data triple

$$(A', B', \mathcal{H}') = ((r)_{i\geq 1}, (s^i)_{i\geq 1}, \{A\} \cup \{B\}_{i\geq 2}).$$

3. A first wild example

In this section we construct our first example of a simple Riesz group (G, G^+) that contains an (even) decreasing sequence of (unbounded) intervals (D_n) such that, the larger n is, the more copies of D_n we have to add in order to get G^+. The main ingredient is the construction carried out in the previous section, which is exploited with a certain recurrence using (infinite) commutative diagrams.

Lemma 3.1. *Let $f\colon (G, G^+) \to (H, H^+)$ be a positive morphism. Let $D \subseteq G^+$ be an interval, and define $D_f = \{x \in H^+ \mid x \leq_H f(y)$ for some $y \in D\}$. Then:*

(i) *D_f is an interval.*
(ii) *If D is countably generated by a sequence (e_n), then D_f is also countably generated, by $(f(e_n))$.*
(iii) *If f is an order-embedding and $tD \neq G^+$ for some t in $\mathbb{N}$, then $tD_f \neq H^+$.*
(iv) *Let $r \in \mathbb{N}$. Assume that D is non-zero, H is simple and f is an order-embedding. If $rD = G^+$ then $rD_f = H^+$.*

Proof. (i) Evidently D_f is non-empty. Let $x \in D_f$ and assume $0 \leq y \leq x$. By construction, there is an element z in D such that $x \leq f(z)$, hence $y \leq f(z)$ and $y \in D_f$. This proves that D_f is order-hereditary. Next, take x, y in D_f. There are then z_1, z_2 in D such that $x \leq f(z_1)$ and $y \leq f(z_2)$. Since D is an interval, there is a z in D such that $z_i \leq z$. Hence $x, y \leq f(z)$, and $f(z) \in D_f$.

(ii) This is trivial.

(iii) Assume that f is an order-embedding. Take x in $G^+ \setminus tD$. Then $f(x) \in H^+ \setminus tD_f$. For, if $f(x) \in tD_f$ there would be an element y in D_f such that $f(x) \le ty$. But then we could find an element d in D such that $y \le f(d)$, hence $f(x) \le f(td)$. Since f is an order-embedding, this yields $x \le td$, a contradiction.

(iv) Let $x \in D_f \setminus \{0\}$, and take z_0 in D such that $x \le f(z_0)$. Since H is simple, we know that x is an order-unit. If now $y \in H^+$, there is n in $\mathbb{N}$ such that $y \le_H nx \le_H nf(z_0) = f(nz_0)$. Now, $nz_0 \in G^+ = rD$, so that we can find z in D for which $nz_0 \le rz$. Thus $y \le_H f(nz_0) \le_H f(rz) = rf(z)$. Hence $y \in rD_f$. □

Definition and Discussion 3.2. Let p and q be positive integers such that $1 < q < p - q$ and that $\gcd(q,p) = 1$. Set $A = \langle q, p-q\rangle$ as in the Definition 2.1. Suppose that (H, H^+) is a simple ordered group of rank one such that there is an order-embedding $(\mathbb{Z}, A) \hookrightarrow (H, H^+)$. By Proposition 1.1, $(H, H^+) = \varinjlim((\mathbb{Z}, G^+_{0,j}), f_{0,j})$ with $(\mathbb{Z}, G^+_{0,0}) = (\mathbb{Z}, A)$ in such a way that $(\mathbb{Z}, G^+_{0,j})$ is a simple component and $f_{0,j} \colon (\mathbb{Z}, G^+_{0,j}) \to (\mathbb{Z}, G^+_{0,j+1})$ is an order-embedding, given by multiplication by a non-negative integer n_j, for all $j \ge 0$. Let $\mathfrak{n} = \prod_{j\ge 0} n_j$ be the generalized integer associated to the sequence (n_j) and assume there exist a positive integer $r > q$ such that r is coprime with $\mathfrak{n}$, and a positive integer s in A such that $\gcd(r,s) = 1$ and $r < s - r$. Put $B = \langle r, s-r\rangle$.

Next, define $G^+_{i,0} = rG^+_{i-1,0} + s^iB$ for $i > 0$, and $G^+_{i,j} = rG^+_{i-1,j} + n_{j-1}G^+_{i,j-1}$ for $i, j > 0$. Let $f_{i,j} \colon (\mathbb{Z}, G^+_{i,j}) \to (\mathbb{Z}, G^+_{i,j+1})$ be the morphism given by multiplication by n_j, and let $g_{i,j} \colon (\mathbb{Z}, G^+_{i,j}) \to (\mathbb{Z}, G^+_{i+1,j})$ be given by multiplication by r. Denote by $G^+_i = G^+_{i,i}$, and $f_i = g_{i,i+1}f_{i,i} = f_{i+1,i}g_{i,i}$.

Define $(K, K^+) = \varinjlim(\mathbb{Z}, G^+_{i,0})$ and notice that, by Remark 2.5, (K, K^+) belongs to the class introduced in [17, Definition 2.1]. It follows then by [17, Proposition 2.5] that (K, K^+) is a simple group of rank one. Observe that this construction yields the following commutative diagram of groups and group morphisms:

$$\begin{array}{ccccccc}
(\mathbb{Z}, G^+_{0,0}) & \xrightarrow{f_{0,0}} & (\mathbb{Z}, G^+_{0,1}) & \xrightarrow{f_{0,1}} & (\mathbb{Z}, G^+_{0,2}) & \xrightarrow{f_{0,2}} & \cdots \\
{\scriptstyle g_{0,0}}\big\downarrow & & \big\downarrow{\scriptstyle g_{0,1}} & & \big\downarrow{\scriptstyle g_{0,2}} & & \\
(\mathbb{Z}, G^+_{1,0}) & \xrightarrow{f_{1,0}} & (\mathbb{Z}, G^+_{1,1}) & \xrightarrow{f_{1,1}} & (\mathbb{Z}, G^+_{1,2}) & \xrightarrow{f_{1,2}} & \cdots \\
{\scriptstyle g_{1,0}}\big\downarrow & & \big\downarrow{\scriptstyle g_{1,1}} & & \big\downarrow{\scriptstyle g_{1,2}} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Proposition 3.3. *For the construction in 3.2, the following conditions hold:*

(i) *$(\mathbb{Z}, G^+_{i,j})$ is a simple component for all choices of i and j.*

(ii) *The morphisms $f_{i,j}$ and $g_{i,j}$ are order-embeddings for all choices of i and j.*

(iii) *Let (G, G^+) be the direct limit of the inductive system $((\mathbb{Z}, G^+_i), f_i)$. Then (G, G^+) is a simple group of rank one and there are order-embeddings from (H, H^+) into (G, G^+) and from (K, K^+) into (G, G^+).*

(iv) *There exists an interval $D_r \subseteq G^+$ such that $tD_r \ne G^+$ for $t \le r - 1$ and $rD_r = G^+$.*

Proof. (i) Since $\gcd(r,s) = 1$, [17, Lemma 2.3 (1)] ensures that $(\mathbb{Z}, G^+_{1,0})$ with $G^+_{1,0} = rA + sB$ is a simple component. Assume now that $(\mathbb{Z}, G^+_{i,0})$ is a simple component. Since $G^+_{i+1,0} =$

$rG^+_{i,0} + s^i B$, we can use [17, Lemma 2.3(1)] again to conclude that $(\mathbb{Z}, G^+_{i+1,0})$ is also a simple component. Hence, by induction $(\mathbb{Z}, G^+_{i,0})$ is a simple component for all choices of i.

Next, assume that for all $k < j$, we have that $(\mathbb{Z}, G^+_{i,k})$ is a simple component for all i. We want to prove that $(\mathbb{Z}, G^+_{i,j})$ is a simple component for all i. By the discussion in 3.2, we know that $(\mathbb{Z}, G^+_{0,j})$ is a simple component. Now, suppose that $(\mathbb{Z}, G^+_{i,j})$ is a simple component for some i. Then, since $G^+_{i+1,j} = rG^+_{i,j} + n_{j-1}G^+_{i+1,j-1}$ and $\gcd(r, n_{j-1}) = 1$, another application of [17, Lemma 2.3(1)] allows us to conclude that $(\mathbb{Z}, G^+_{i+1,j})$ is also a simple component. The proof is then complete by induction.

(ii) By assumption, $f_{0,j}$ is an order embedding for all j. Notice also that $g_{i,0}$ is also an order-embedding for all i, by Proposition 2.4. Assume that $f_{i,j}$ is an order embedding, and consider the following diagram:

$$\begin{array}{ccc}
(\mathbb{Z}, G^+_{i,j}) & \xrightarrow{f_{i,j}=n_j\cdot} & (\mathbb{Z}, G^+_{i,j+1}) \\
{\scriptstyle g_{i,j}=r\cdot}\big\downarrow & & \big\downarrow{\scriptstyle g_{i,j+1}=r\cdot} \\
(\mathbb{Z}, G^+_{i+1,j}) & \xrightarrow{f_{i+1,j}=n_j\cdot} & (\mathbb{Z}, G^+_{i+1,j+1})
\end{array}$$

Since $G^+_{i+1,j+1} = f_{i+1,j}(G^+_{i+1,j}) + g_{i,j+1}(G^+_{i,j+1})$, we conclude from [17, Proposition 2.10] that $f_{i+1,j}$ and $g_{i,j+1}$ are also an order-embeddings. Hence, it follows by induction that $f_{i,j}$ and $g_{i,j}$ are order-embeddings for all choices of i and j.

(iii) That (G, G^+) is a simple group follows from [17, Lemma 2.4]. By [16, Lemma 2.4], G is isomorphic to $\mathbb{Z}_{\mathfrak{n}r^\infty}$, and so it is a group of rank one.

For every j, let $g_j \colon (\mathbb{Z}, G^+_{0,j}) \to (\mathbb{Z}, G^+_j)$ be defined by $g_j = g_{j-1,j} \cdots g_{1,j} g_{0,j}$. Then g_j is an order-embedding and $g_{j+1} f_{0,j} = f_j g_j$ for all j. It follows then from [17, Lemma 2.9] that the naturally induced map $\varphi \colon (H, H^+) \to (G, G^+)$ is an order-embedding. In a similar fashion, there is an order-embedding $\psi \colon (K, K^+) \to (G, G^+)$.

(iv) By Proposition 2.4, together with Lemma 2.2 and Proposition 2.3, K^+ contains an interval D such that $tD \neq K^+$ for $t \leq r-1$ and $rD = K^+$. Therefore, if we let $D_r = D_\psi$, then conditions (iii) and (iv) in Lemma 3.1 ensure that D_r will do the job. $\square$

Theorem 3.4. *Let $J = (q_i)_{i \geq 1}$ be a sequence of non-negative, relatively prime integers. Let $I = (a_j)_{j \geq 1}$ be a sequence such that every $a_j \in J$, while each q_i in J appears infinitely many times in I. Let $\mathfrak{n} = \prod_{k \geq 1} q_k^\infty$ be the generalized integer associated to J. Then, for any infinite generalized integer $\mathfrak{m}$ that is coprime with $\mathfrak{n}$, there exists a simple Riesz group of rank one $G(\mathfrak{m})$ such that:*

(i) *For every $q_i \in J$, there is a countably generated interval D_i satisfying $tD_i \neq G(\mathfrak{m})^+$ for $t \leq q_i - 1$ and $q_i D_i = G(\mathfrak{m})^+$.*
(ii) *The group $G(\mathfrak{m})$ is isomorphic to $\mathbb{Z}_{\mathfrak{n}\cdot\mathfrak{m}}$ (as abelian groups).*

Proof. We first construct a simple ordered group of rank one (G, G^+) satisfying condition (i) and such that $G \cong \mathbb{Z}_{\mathfrak{n}}$. To do so, we proceed inductively.

Take p_1 such that $p_1 > 2q_1$ and $\gcd(p_1, q_1) = 1$. Let $A = \langle q_1, p_1 - q_1 \rangle$. Take p_2 in A such that $p_2 > 2q_2$ and $\gcd(p_2, q_2) = 1$. Let $B_1 = \langle q_2, p_2 - q_2 \rangle$. We construct the following commutative diagram of groups and group morphisms:

$$
\begin{tikzcd}
(\mathbb{Z}, G^{(1)+}_{0,0}) \arrow[r, "q_1\cdot"] \arrow[d, "q_2\cdot"'] & (\mathbb{Z}, G^{(1)+}_{0,1}) \arrow[r, "q_1\cdot"] \arrow[d, "q_2\cdot"] & (\mathbb{Z}, G^{(1)+}_{0,2}) \arrow[r, "q_1\cdot"] \arrow[d, "q_2\cdot"] & \cdots \\
(\mathbb{Z}, G^{(1)+}_{1,0}) \arrow[r, "q_1\cdot"] \arrow[d, "q_2\cdot"'] & (\mathbb{Z}, G^{(1)+}_{1,1}) \arrow[r, "q_1\cdot"] \arrow[d, "q_2\cdot"] & (\mathbb{Z}, G^{(1)+}_{1,2}) \arrow[r, "q_1\cdot"] \arrow[d, "q_2\cdot"] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
\tag{3.1}
$$

where $G^{(1)+}_{0,0} = A$, $G^{(1)+}_{0,i} = q_1 G^{(1)+}_{0,i-1} + p_1^i A$, $G^{(1)+}_{i,0} = q_2 G^{(1)+}_{i-1,0} + p_2^i B_1$. By [17, Lemma 2.3], $(\mathbb{Z}, G^{(1)+}_{0,i})$ is a simple component for all i and the maps in the top row are order-embeddings. By Proposition 3.3, all groups $(\mathbb{Z}, G^{(1)+}_{i,j})$ are simple components and all maps in the diagram are order-embeddings.

Let (G_0, G_0^+) be the direct limit of the top row, (H_0, H_0^+) be the limit of the first column, and let (G_1, G_1^+) be the limit of the diagonal terms (under the natural maps, obtained by composition). By [17, Proposition 2.5], (G_0, G_0^+) and (H_0, H_0^+) are simple groups of rank one. By condition (iii) in Proposition 3.3, (G_1, G_1^+) is also a simple group of rank one and there are order-embeddings

$$\tau_0\colon (G_0, G_0^+) \to (G_1, G_1^+)\,, \text{ and } \psi_0\colon (H_0, H_0^+) \to (G_1, G_1^+)\,.$$

By Lemma 2.2, Proposition 2.3 and Proposition 2.4, there are countably generated intervals D_1' in G_0^+ and D^0 in H_0^+ such that $tD_1' \neq G_0^+$ if $t \leq q_1 - 1$ and $q_1 D_1' = G_0^+$; also, $tD^0 \neq H_0^+$ if $t \leq q_2 - 1$, and $q_2 D^0 = H_0^+$.

Let $D_2' = (D^0)_{\psi_0}$. Then, Lemma 3.1 ensures that D_2' is a countably generated interval in G_1^+ such that $tD_2' \neq G_1^+$ for $t \leq q_2 - 1$, and $q_2 D_2' = G_1^+$.

Next, relabel the diagonal as the top row (i.e. let $G^{(2)+}_{0,i} = G^{(1)+}_{i,i}$ for $i \geq 0$) and take p_3 in A such that $p_3 > 2q_3$ and $\gcd(p_3, q_3) = 1$. Let $B_2 = \langle q_3, p_3 - q_3 \rangle$, and construct a commutative diagram as before:

$$
\begin{tikzcd}
(\mathbb{Z}, G^{(2)+}_{0,0}) \arrow[r, "q_1q_2\cdot"] \arrow[d, "q_3\cdot"'] & (\mathbb{Z}, G^{(2)+}_{0,1}) \arrow[r, "q_1q_2\cdot"] \arrow[d, "q_3\cdot"] & (\mathbb{Z}, G^{(2)+}_{0,2}) \arrow[r, "q_1q_2\cdot"] \arrow[d, "q_3\cdot"] & \cdots \\
(\mathbb{Z}, G^{(2)+}_{1,0}) \arrow[r, "q_1q_2\cdot"] \arrow[d, "q_3\cdot"'] & (\mathbb{Z}, G^{(2)+}_{1,1}) \arrow[r, "q_1q_2\cdot"] \arrow[d, "q_3\cdot"] & (\mathbb{Z}, G^{(2)+}_{1,2}) \arrow[r, "q_1q_2\cdot"] \arrow[d, "q_3\cdot"] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
\tag{3.2}
$$

Observe that, by construction, (G_1, G_1^+) is the inductive limit of the first row. Let (H_1, H_1^+) be the inductive limit of the first column and (G_2, G_2^+) the inductive limit of the diagonal terms. The same line of argument as before shows that (H_1, H_1^+) and (G_2, G_2^+) are simple groups of rank one and that there are order-embeddings

$$\tau_1\colon (G_1, G_1^+) \to (G_2, G_2^+)\,, \text{ and } \psi_1\colon (H_1, H_1^+) \to (G_2, G_2^+)\,.$$

Another application of Lemma 2.2, Proposition 2.3 and Proposition 2.4 provides us with a countably generated interval D^1 in H_1^+ such that $tD^1 \neq H_1^+$ if $t \leq q_3 - 1$ and $q_3 D^1 = H_1^+$.

Let $D_3' = (D^1)_{\psi_1}$. Then D_3' is also a countably generated interval (in G_2^+), by Lemma 3.1, that satisfies $tD_3' \neq G_2^+$ for $t \leq q_3 - 1$ and $q_3 D_3' = G_2^+$.

Continuing in this way, we get a sequence of simple groups of rank one and order-embeddings

$$(G_0, G_0^+) \xrightarrow{\tau_0} (G_1, G_1^+) \xrightarrow{\tau_1} (G_2, G_2^+) \xrightarrow{\tau_2} \cdots$$

such that for each i, G_i^+ contains a countably generated interval D'_{i+1} such that $tD'_{i+1} \neq G_i^+$ for $t \leq q_{i+1} - 1$ and $q_{i+1} D'_{i+1} = G_i^+$.

Let (G, G^+) be the limit of this inductive system. Denote by $\overline{\tau}_i \colon (G_i, G_i^+) \to (G, G^+)$ the natural maps. Now define $D''_{i+1} = (D'_{i+1})_{\overline{\tau}_i}$. By Lemma 3.1, all the intervals D''_j will satisfy $tD''_j \neq G^+$ for all $t \leq q_j$, and $q_j D''_j = G^+$. By [17, Lemma 2.4], (G, G^+) is a simple group, and since $G \cong \mathbb{Z}_{\mathfrak{n}}$ by construction, it is a group of rank one.

Now, given any infinite generalized integer $\mathfrak{m}$ coprime with $\mathfrak{n}$, Theorem 1.2 ensures the existence of a simple Riesz group of rank one $(G(\mathfrak{m}), G(\mathfrak{m})^+)$ and an order embedding $\tau \colon (G, G^+) \to (G(\mathfrak{m}), G(\mathfrak{m})^+)$ such that $G(\mathfrak{m})$ is isomorphic to $\mathbb{Z}_{\mathfrak{n} \cdot \mathfrak{m}}$ (as abelian groups), thus proving condition (ii). For each $i \geq 1$ define $D_i = (D''_i)_\tau$. Then, by Lemma 3.1, for every q_i in I, D_i satisfies that $tD_i \neq G(\mathfrak{m})^+$ for $t \leq q_i - 1$ and $q_i D_i = G(\mathfrak{m})^+$. $\square$

Let (G, u) be a partially ordered abelian group with order-unit. We denote by $S(G, u)$ (or by S_u if no confusion may arise) the compact convex space of *states* on (G, u), that is, the set of group morphisms $s \colon G \to \mathbb{R}$ such that $s(u) = 1$. We use $\mathrm{Aff}(S_u)^+$ to refer to the monoid of positive, affine and continuous functions from S_u to $\mathbb{R}^+$, and $\phi_u \colon G^+ \to \mathrm{Aff}(S_u)^+$ stands for the natural evaluation map. Let $\mathrm{LAff}_\sigma(S_u)^{++}$ be the monoid of strictly positive, affine, lower semicontinuous functions from G^+ to $\mathbb{R}^+$ that are pointwise suprema of increasing sequences of functions from $\mathrm{Aff}(S_u)^+$.

If D is a fixed interval in $\Lambda_\sigma(G^+)$, we denote by $\Lambda_{\sigma,D}(G^+)$ the submonoid of $\Lambda_\sigma(G^+)$ whose elements are intervals X in $\Lambda_\sigma(G^+)$ such that $X \subseteq nD$ for some n in $\mathbb{N}$, and we denote by $W_\sigma^D(G^+)$ the submonoid of $\Lambda_{\sigma,D}(G^+)$ whose elements are intervals X in $\Lambda_{\sigma,D}(G^+)$ such that there exists Y in $\Lambda_{\sigma,D}(G^+)$ with $X + Y = nD$ for some n in $\mathbb{N}$. If now D is a countably generated interval in G^+ that is also generating, set $d = \sup \phi_u(D)$, and define (see [19])

$$W_\sigma^d(S_u) = \{f \in \mathrm{LAff}_\sigma(S_u)^{++} \mid f + g = nd \text{ for some } g \text{ in } \mathrm{LAff}_\sigma(S_u)^{++} \text{ and } n \text{ in } \mathbb{N}\}.$$

The disjoint union $G^+ \sqcup W_\sigma^d(S_u)$ can be endowed with a monoid structure by extending the natural operations and setting $x + f = \phi_u(x) + f$ whenever $x \in G^+$ and $f \in W_\sigma(S_u)$.

Recall that an interval X in G^+ is said to be *soft* (see, e.g. [9]) provided that for each x in X, there is an element y in X and a natural number n such that $(n+1)x \leq_G ny$. Observe that in case the interval X satisfies $rX = G^+$, then X is soft. Indeed, if $x \in X \setminus \{0\}$, then $(r+1)x \in G^+ = rX$, hence there is an element y in X such that $(r+1)x \leq_G ry$.

It was proved in [19, Theorem 3.8] that, if (G, u) is a simple Riesz group with order-unit, and D is a non-zero, soft, countably generated interval in G^+, then the map

$$\varphi \colon W_\sigma^D(G^+) \to G^+ \sqcup W_\sigma^d(S_u) \tag{3.3}$$

given by the rule $\varphi([0, x]) = x$ for any x in G^+, and by $\varphi(X) = \sup \phi_u(X)$ for any soft interval X in $W_\sigma^D(G^+)$, is a normalized monoid morphism. It becomes an isomorphism if G satisfies some additional assumptions, namely if G is non-atomic and strictly unperforated. Recently, the first and second authors have shown that injectivity is equivalent to strict unperforation [14, Theorem 3.2], and surjectivity corresponds to a special property satisfied by the generating interval D [14, Theorem 3.5]. If D is a soft generating interval such that $\varphi(D)$ is identically infinite, then we say that a soft interval X in $W_\sigma^D(G^+)$ is *unbounded* provided that $\varphi(X) = \sup \phi_u(X) = \infty$. Notice that this does not depend on the choice of the order-unit. If v is another order-unit for G, then it follows from [8, Proposition 6.17] that the state spaces S_u and S_v are homeomorphic.

For the proof in the result below, we recall the following definition: An abelian monoid M is a *refinement monoid* if, for all x_1, x_2, y_1, y_2 in M that satisfy $x_1 + x_2 = y_1 + y_2$, there exist elements z_{ij} in M, for $i, j = 1, 2$, such that $\sum_{j=1}^{2} z_{ij} = x_i$ and $\sum_{i=1}^{2} z_{ij} = y_j$ (see, e.g. [28]).

Proposition 3.5. *Let (G, G^+) be a simple Riesz group, let $I = (q_i)_{i\geq 1}$ be an increasing sequence of non-negative integers such that $\gcd(q_i, q_j) = 1$ for all different i and j. For every q_i in I, assume that D_i is a countably generated interval in $W_\sigma^D(G^+)$ satisfying $tD_i \neq G^+$ for $t \leq q_i - 1$ and $q_iD_i = G^+$. Then there exists a descending sequence of intervals (X_i) such that $tX_i \neq G^+$ for $t \leq q_i - 1$, and $(\prod_{j=1}^{i} q_j)X_i = G^+$.*

Proof. Let $M = \Lambda_\sigma(G^+)$ be the monoid of countably generated intervals in G^+, with the algebraic ordering that we shall denote by $\leq_M$. By [11, Proposition 2.5], $\Lambda_\sigma(G^+)$ is a refinement monoid. Let $X_1 = D_1$. Since $q_2D_2 = G^+$, we have $X_1 + G^+ = q_2D_2$. Hence, by [28, Lemma 1.9], there exist intervals $X_{11}, X_{12}, ..., X_{1q_2} \leq_M D_2$ such that $X_1 = X_{11} + X_{12} + ... + X_{1q_2}$ and $X_{11} \leq_M X_{12} \leq_M \cdots \leq_M X_{1q_2} \leq_M X_1$. Let $X_2 = X_{1q_2}$. Notice that, if $tX_2 = G^+$ for any $t \leq q_2 - 1$, then $tD_2 = G^+$, contradicting our assumption on D_2. Thus $tX_2 \neq G^+$ for $t \leq q_2 - 1$. Observe that

$$X_1 = X_{11} + \cdots + X_{1q_2} \leq_M X_{1q_2} + \cdots + X_{1q_2} = q_2X_2\,.$$

Since $q_1X_1 = q_1D_1 = G^+$, we have that $(q_1q_2)X_2 = G^+$. Now we can apply the same argument to the equality $X_2 + G^+ = q_3D_3$, so that we get an interval $X_3 \leq_M X_2$ such that $tX_3 \neq G^+$ for $t \leq q_3 - 1$ and $(q_1q_2q_3)X_3 = G^+$. Continuing in this way we get a descending sequence of intervals $(X_i)_{i\geq 1}$ such that $tX_i \neq G^+$ for $t \leq q_i - 1$ and $(\prod_{j=1}^{i} q_j)X_i = G^+$. □

Remark 3.6. Notice that, if (G, G^+, u) is a partially ordered abelian group with order-unit, and $D \subseteq G^+$ is an interval such that $nD = G^+$ for some natural number n, then $\varphi(D) = \infty$, i.e. D is an unbounded interval. To see this, notice that, given any non-zero element x in G^+, there exists an element y in D such that $x \leq_G ny$. Hence, for any state s on G we have $0 \leq s(x) \leq ns(y)$, i.e. $0 \leq \frac{\phi_u(x)}{n} \leq \phi_u(y)$. Thus, in order to see that $\varphi(D) = \infty$, it is enough to show that $\varphi(G^+) = \infty$. But now, for every m in $\mathbb{N}$, we have that $mu \in G^+$, and then $0 < m = \phi_u(mu)$, whence $\varphi(G^+) = \infty$. In particular, this fact applies to the intervals D_i, X_j in Proposition 3.5.

The construction just carried out in Theorem 3.4 guarantees that we are in position to apply Proposition 3.5 and obtain a somewhat more refined example, as follows.

Theorem 3.7. *Let $J = (q_i)_{i\geq 1}$ be a sequence of non-negative integers such that $\gcd(q_i, q_j) = 1$ for all $i, j \geq 1$ (such that $i \neq j$), and let $I = (a_j)_{j\geq 1}$ be a sequence such that every $a_j \in J$, while each $q_i \in J$ appears an infinite number of times in I. Let $\mathfrak{m}$ be a generalized integer, that is coprime with $\mathfrak{n}(I)$. Let $(G, G^+) = (G(\mathfrak{m}), G^+(\mathfrak{m}))$ be the simple Riesz group of rank one constructed in Theorem 3.4. Then G contains a descending sequence of unbounded intervals (D_i) such that $(\prod_{j=1}^{i} q_j)D_i = G^+$ for all i, while $tD_i \neq G^+$ whenever $t \leq q_i - 1$.*

Proof. We only need to check that (G, G^+) fulfills the hypotheses of Proposition 3.5. The sequence of intervals obtained in the conclusion of Theorem 3.4, say (D'_i), satisfies that $tD'_i \neq G^+$ for all $t \leq q_i - 1$, and $q_i D'_i = G^+$ for all i. Thus, result holds by Proposition 3.5. □

The examples we have just obtained could be considered as an intermediate step towards constructing a simple Riesz group (G, G^+), together with an unbounded interval D in G^+ such that $nD \neq G^+$ for every n in $\mathbb{N}$. In fact, a natural candidate for such an interval could the intersection of the descending chain of intervals appearing in Theorem 3.7. Unfortunately, even under the hypotheses of Theorem 3.7, we are not able to prove whether or not the intersection $D = \bigcap_{i \geq 1} D_i$ is an interval or even an unbounded subset of G^+, where (D_i) is a descending sequence of countably generated, unbounded soft intervals (D_i) such that $tD_i \neq G^+$ for $t \leq q_i - 1$ and $(\prod_{j=1}^{i} q_j) D_i = G^+$ (for every $i \geq 1$).

4. Taylor-made gaps in simple components under order-embeddings

In order to obtain our desired example (announced in the Introduction) of a simple Riesz group of rank one (G, G^+), together with an unbounded (countably generated) interval D in G^+ such that $nD \neq G^+$ for all n, we adopt the basic philosophy of [16, Section 3]. This consists of reducing the essential properties that an interval should have to a finite set of properties occurring in simple components. For this, we need to have some control over those non-negative integers in a simple component that its positive cone may contain, and we also need to construct order-embeddings among simple components under which this control is preserved.

In view of the considerations, made at the beginning of Section 1 and related to results on simple components (see [16] and [24]), a possible way of getting the desired control is to consider submonoids of the non-negative integers generated by coprime positive integers, and order-embeddings among simple components whose positive cones have this particular form, using [17, Lemma 2.3 (2)]. The basic idea consists of strengthening some arithmetic properties in order to force the expression of non-negative integers to become positive elements in a certain simple component.

Lemma 4.1. *Let $N \in \mathbb{N}$. Given a in $\mathbb{N}$, there exist p, c and d in $\mathbb{N}$ such that* $\gcd(a, p) = \gcd(a, c) = \gcd(c, d) = 1$, *$pc \equiv pd \equiv 1 \pmod{a}$, $p > N$, $pc > aN$ and $d > \max\{(a-1)pc + a(N-1), ac\}$.*

Proof. Throughout the proof, denote by $\overline{x}$ the class of an element in $\mathbb{Z}/n\mathbb{Z}$ for any n. For p and c in $\mathbb{N}$, it is clear that $\gcd(p, a) = \gcd(c, a) = 1$ is equivalent to the fact that $\overline{p}$ and $\overline{c}$ are invertible in $\mathbb{Z}/a\mathbb{Z}$. Therefore, if we take p and c in such a way that $\overline{p} = \overline{c}^{-1} \in \mathbb{Z}/a\mathbb{Z}$, we will have that $pc \equiv 1 \pmod{a}$. It is clear that there exist infinitely many numbers p and c satisfying the above. We can then take p, $c > N$ and also $pc > aN$. By a similar line of argument, once p is fixed, there are infinitely many d in $\mathbb{N}$ such that $pd \equiv 1 \pmod{a}$. For any one of this choices we have that $\overline{d} = \overline{p}^{-1} = \overline{c}$ in $\mathbb{Z}/a\mathbb{Z}$, whence $d - c$ is divisible by a, that is $d = c + ak$ for some k in $\mathbb{N}$. Now, in $\mathbb{Z}/c\mathbb{Z}$, this says $\overline{d} = \overline{c} + \overline{a}\overline{k} = \overline{a}\overline{k}$. We can choose k big enough such that $d > \max\{(a-1)pc + a(N-1), ac\}$ and $\gcd(c, k) = 1$. This will also guarantee that d is invertible in $\mathbb{Z}/c\mathbb{Z}$), that is $\gcd(c, d) = 1$. □

Notation. Let $(\mathbb{Z}, H^+)$ be a simple component. There is then a (uniquely determined) element N in H^+ such that $N-1 \notin H^+$, and $N+k \in H^+$ for all k in $\mathbb{Z}^+$. We shall denote this element by N_H.

For the rest of this section, let us fix a simple component $(\mathbb{Z}, H^+)$. Given a in $\mathbb{N}$ we can choose by Lemma 4.1 elements p, c and d in $\mathbb{N}$ such that $\gcd(a,p) = \gcd(a,c) = \gcd(c,d) = 1$, $pc \equiv pd \equiv 1 \pmod{a}$, $p \in H^+$, $pc > aN_H$ and $d > \max\{(a-1)pc + a(N_H - 1), ac\}$.

Let $G^+ = aH^+ + p\langle c, d\rangle$. Since $\gcd(c,d) = 1$, we have that $(\mathbb{Z}, \langle c, d\rangle)$ is a simple component. Since $\gcd(a,p) = 1$, we can use [17, Lemma 2.3] and conclude that $(\mathbb{Z}, G^+)$ is a simple component and the map $(\mathbb{Z}, H^+) \to (\mathbb{Z}, G^+)$ defined by multiplication by a is an order-embedding.

We shall use these notations in the Proposition below and in the next section.

Proposition 4.2. *Let i in $\mathbb{Z}$ be such that $0 \le i \le a-1$ and let $x \notin H^+$. Then $ipc + ax \notin G^+$. In particular, if we denote $L_H = \{l_0, l_1, \dots l_{a-1}\}$ where $l_i = ipc + a(N_H - 1)$, it follows that $L_H \cap G^+ = \emptyset$. Moreover, all integers that are congruent to $i \pmod{a}$ and bigger than l_i belong to G^+.*

Proof. Since multiplication by a is an order-embedding, we see that $at \notin G^+$ if and only if $t \notin H^+$. In particular, $a(N_H - 1) \notin G^+$. Moreover, any multiple of a which is bigger than $a(N_H - 1)$ will belong to G^+, as it will have the form at with $t = N_H + k$ (k in $\mathbb{Z}^+$), and so $t \in H^+$.

Assume now that we have $0 \le i \le a-1$ and $x \notin H^+$. We have to prove that $ipc + ax \notin G^+$. By way of contradiction, if $ipc + ax \in G^+$, we would then have that $ipc + ax = ay + pz$ where $y \in H^+$ and $z \in \langle c, d\rangle$, that is, $z = cz_1 + dz_2$ for some positive integers z_1, z_2. We therefore have

$$ipc + ax = ay + pcz_1 + pdz_2\,. \tag{4.1}$$

We now claim that $y < x$. We already know that $y \neq x$ because $y \in H^+$. Assume that $y \ge x+1$, so that $y = x+k$ with $k \ge 1$. We would then have that $ipc+ax = ay+pcz_1+pdz_2 = ax + ak + pcz_1 + pdz_2$, whence $ipc = ak + pcz_1 + pdz_2$. Thus (since also $c < d$),

$$0 < a \le ak = ipc - pcz_1 - pdz_2 \le ipc - pcz_1 - pcz_2 = pc(i - (z_1 + z_2))\,,$$

and so $i - (z_1 + z_2) \ge 0$. Notice that also

$$0 \equiv ak = ipc - (pcz_1 + pdz_2) \equiv i - (z_1 + z_2) \pmod{a}\,.$$

This implies that $i - (z_1 + z_2) = ar$ for some positive integer r. But since $i \le a-1$ by assumption we conclude that $r = 0$, hence $i = z_1 + z_2 \le a-1$. Then $ipc = ak + pcz_1 + pdz_2 > ak + pc(z_1 + z_2) = ak + pci$, and so $0 > ak > 0$. This contradiction establishes the claim.

Going back to equation (4.1), we find that $0 \le a(x-y) = pcz_1 + pdz_2 - ipc$. Since $x \notin H^+$, we have that $y < x \le N_H - 1$. If $z_2 \neq 0$, then $az_2 - (a-1) \ge 1$ and (using that $d > ac$ and that $pc > aN_H$), we get

$$\begin{aligned} a((N_H - 1) - y) &\ge a(x-y) = pcz_1 + pdz_2 - ipc \\ &\ge pdz_2 - ipc > pacz_2 - ipc \ge pacz_2 - (a-1)pc \\ &= pc(az_2 - (a-1)) > aN_H(az_2 - (a-1)) > aN_H\,, \end{aligned}$$

which is clearly not possible.

It follows then that $y_2 = 0$. This means that $a(x-y) = pcz_1 - ipc = pc(z_1 - i)$, whence $z_1 > i$ (because $x > y$). But then $a(x-y) = pc(z_1 - i) \geq pc > aN_H$. This implies that $x - y \in H^+$ and since $y \in H^+$, it follows that $x \in H^+$, contrary to our assumption.

We have proved that $ipc + ax \notin G^+$, whenever $0 \leq i \leq a-1$ and $x \notin H^+$. In particular, since $N_H - 1 \notin H^+$, we have that $l_i = ipc + a(N_H - 1) \notin G^+$, hence $L_H \cap G = \emptyset$.

Let now t in $\mathbb{Z}^+$ be such that $t \equiv i \,(\mathrm{mod}\; a)$ and $t > ipc + a(N_H - 1)$. Then, since $pc \equiv 1 \,(\mathrm{mod}\; a)$, we have that $t - ipc \equiv 0 \,(\mathrm{mod}\; a)$ and so $a(N_H - 1) < t - ipc = as$ for some s in $\mathbb{Z}^+$. Then $N_H - 1 < s$, and therefore $s \in H^+$ and $t = as + ipc \in aH^+ + p\langle c, d\rangle = G^+$. It follows from this that any integer congruent to i (mod a) which is bigger than l_i can be written as $ipc + as$ where $s > N_H - 1$ and so they all belong to G^+. □

Corollary 4.3. *Under the hypotheses and notation of Proposition 4.2, we have* $N_G = l_{a-1} + 1$.

Proof. It is clear that l_{a-1} is the largest element of the set L_H defined in Proposition 4.2. Let $x > l_{a-1}$. We obviously have that $x \equiv k \,(\mathrm{mod}\; a)$, for some $0 \leq k \leq a-1$. Since $x > l_{a-1} > l_k$ we obtain (using Proposition 4.2) that $x \in G^+$. □

5. A new wild example

The main objective of this section is to construct a simple Riesz group (G, G^+) of rank one such that its positive cone contains an unbounded interval D that satisfies $nD \neq G^+$ for all n in $\mathbb{N}$. This will be done inductively by constructing a sequence of simple components connected by order-embeddings. We first establish a Lemma that will provide the inductive step in the Theorem below. Given a simple component $(\mathbb{Z}, H^+)$, retain from the previous section the notation N_H for the (unique) element in H^+ such that $N_H - 1 \notin H^+$ but $N_H + k \in H^+$ for all positive integers k.

Lemma 5.1. *Let $(\mathbb{Z}, H^+)$ be a simple component, let x_1, $y_1 \in H^+$ be such that $y_1 = x_1 + 1$, and let $a > N_H$. There exists then a simple component $(\mathbb{Z}, G^+)$ satisfying:*

(i) $a\cdot : (\mathbb{Z}, H^+) \to (\mathbb{Z}, G^+)$ *is an order-embedding and* $a^2 N_H < N_G$.

(ii) *There is an element y_2 in G^+ such that:*

  (a) $y_2 - 1 \in G^+$.

  (b) $ay_1 <_G y_2$ *and* $y_2 - ay_1 > aN_H$.

  (c) $(N_H - 1)ax_1 \not<_G (N_H - 1)y_2$.

Proof. Notice that $(N_H - 1)y_1 - (N_H - 1)x_1 = (N_H - 1) \notin H^+$, whence $(N_H - 1)x_1 \not\leq_H (N_H - 1)y_1$.

Choose p, c and d as in Lemma 4.1. Letting $G^+ = aH^+ + p\langle c, d\rangle$, we have that $(\mathbb{Z}, G^+)$ is a simple component and multiplication by a is an order-embedding.

Write $L_H = \{l_0, l_1, \ldots, l_{a-1}\}$ as in Proposition 4.2, so we have that any integer congruent to i (mod a) which is larger than l_i belongs to G^+. Note also that $N_G = l_{a-1} + 1$, by Corollary 4.3. This equals to $N_G = (a-1)pc + a(N_H - 1) + 1$, and hence we have

$$\begin{aligned} N_G &= (a-1)pc + a(N_H - 1) + 1 > (a-1)(pc + (N_H - 1)) \\ &> (a-1)(aN_H + (N_H - 1)) > (a-1)(a+1)(N_H - 1) \\ &= (a^2 - 1)(N_H - 1) > a^2 N_H\,, \end{aligned}$$

proving condition (i).

Let $y_2 = pc + ay_1$, and observe that $y_2 \in G^+$, because $y_1 \in H^+$ by assumption. Notice also that $y_2 - ay_1 = pc > aN_H$, by the election of p and c. Since $pc \in G^+$, we see that $ay_1 <_G y_2$, thus verifying condition (ii)(b).

Since $a > N_H$, it follows from Proposition 4.2 that $l_{N_H-1} \notin G^+$. Therefore, the fact that

$$(N_H - 1)y_2 - (N_H - 1)ax_1 = (N_H - 1)pc + a(N_H - 1) = l_{N_H-1} \notin G^+$$

implies that $(N_H - 1)ax_1 \nleq_G (N_H - 1)y_2$. Hence condition (ii)(c) also holds.

It remains to verify condition (ii) (a). Since $y_2 - 1 = pc + ay_1 - 1 \equiv 0 \pmod a$ and $y_2 - 1 = pc + ay_1 - 1 \geq pc > aN_H > a(N_H - 1) = l_0$, Proposition 4.2 ensures that $y_2 - 1 \in G^+$, as desired. □

Theorem 5.2. *Let A be a strictly ascending sequence of non-negative integers. Then, for any generalized integer $\mathfrak{m}$ coprime with $\mathfrak{n}(A)$, there exists a simple Riesz group of rank one $G(\mathfrak{m})$ such that:*

(i) *There is an unbounded countably generated interval D satisfying $nD \neq G(\mathfrak{m})^+$ for all n in $\mathbb{N}$.*
(ii) *For some generalized integer $\mathfrak{n}$ dividing $\mathfrak{n}(A)$, the group $G(\mathfrak{m})$ is isomorphic to $\mathbb{Z}_{\mathfrak{n}\mathfrak{m}}$ (as abelian groups).*

Proof. First, we will inductively construct a sequence of simple components and order-embeddings

$$a_i\cdot : (\mathbb{Z}, H_i^+) \to (\mathbb{Z}, H_{i+1}^+)\,,$$

together with a sequence (y_i) in $\mathbb{Z}^+$ $(i \geq 1)$ such that

(a) For all $i \geq 1$, $a_i \in A$.
(b) For all $i \geq 1$, $a_i > N_{H_i} > a_{i-1}^2 N_{H_{i-1}} > (a_1^2)^{i-1} N_{H_1}$ for all $i \geq 1$. Also $y_i \in H_i^+$, and the element $x_i = y_i - 1 \in H_i^+$ for all i.
(c) $(N_{H_i} - 1)x_i \not<_{H_i} (N_{H_i} - 1)y_i$.
(d) $a_i y_i <_{H_{i+1}} y_{i+1}$.
(e) $(N_{H_j} - 1)a_{i-1}a_{i-2}\cdots a_j x_j \nleq_{H_i} (N_{H_j} - 1)y_i$ for all $j \leq i - 1$.

Let $(\mathbb{Z}, H_1^+)$ be any simple component such that $1 \notin H_1^+$. Let y_1 in H_1^+ be such that $x_1 = y_1 - 1 \in H_1^+$ (for example, $y_1 = N_{H_1}+1$). Choose a_1 in A with $a_1 > \max\{N_{H_1}, 3\}$. Then Lemma 5.1 provides us with a simple component $(\mathbb{Z}, H_2^+)$ (where $N_{H_2} > a_1^2 N_{H_1}$) and an element $y_2 \in H_2^+$ such that the element $x_2 = y_2 - 1 \in H_2^+$, $a_1 y_1 <_{H_2} y_2$, $(N_{H_1} - 1)a_1 x_1 \not<_{H_1} (N_{H_1} - 1)y_2$ and $a_1 N_{H_1} < y_2 - a_1 y_1$. Moreover, multiplication by a_1 is an order-embedding from $(\mathbb{Z}, H_1^+)$ into $(\mathbb{Z}, H_2^+)$.

Suppose that $a_1, \ldots, a_{n-1}$, $H_1^+, \ldots, H_n^+$ and $y_1, \ldots, y_n$ have been constructed satisfying conditions (a)-(e) above.

Choose a_n in A with $a_n > N_{H_n}$, and apply Lemma 5.1 to obtain an order-embedding

$$a_n\cdot : (\mathbb{Z}, H_n^+) \to (\mathbb{Z}, H_{n+1}^+)\,,$$

where $(\mathbb{Z}, H_{n+1}^+)$ is a simple component, such that $N_{H_{n+1}} > a_n^2 N_{H_n}$. Moreover, there is an element y_{n+1} in H_{n+1}^+ such that the element $x_{n+1} = y_{n+1} - 1 \in H_{n+1}^+$, $a_n y_n <_{H_{n+1}} y_{n+1}$, $y_{n+1} - a_n y_n > a_n N_{H_n}$ and $(N_{H_n} - 1)a_n x_n \not<_{H_{n+1}} (N_{H_n} - 1)y_{n+1}$. Hence conditions (a)-(d) are satisfied (as well as condition (e) with $i = n + 1$ and $j = n$).

Thus, in order to see that condition (e) also holds with $i = n+1$, we only need to consider the cases where $j \leq n - 1$.

Notice that $y_{n+1} = p_n c_n + a_n y_n$ (by the proof of Lemma 5.1) where p_n, c_n are chosen as in Lemma 4.1. By our induction hypothesis,

$$(N_{H_j} - 1)y_n - (N_{H_j} - 1)\prod_{k=j}^{n-1} a_k x_j \notin H_n^+ ,$$

whenever $j \leq n-1$.

Since $N_{H_j} - 1 < a_n - 1$ if $j \leq n-1$, Proposition 4.2 applies and so

$$(N_{H_j} - 1)p_n c_n + a_n[(N_{H_j} - 1)y_n - (N_{H_j} - 1)\prod_{k=j}^{n-1} a_k x_j] \notin H_{n+1}^+ ,$$

that is,

$$(N_{H_j} - 1)y_{n+1} - (N_{H_j} - 1)\prod_{k=j}^{n} a_k x_j \notin H_{n+1}^+$$

for every $j \leq n-1$, as desired.

Next, let $(G, G^+) = \varinjlim((\mathbb{Z}, H_i^+), a_i \cdot)$, and denote by $f_n \colon (\mathbb{Z}, H_n^+) \to (G, G^+)$ the natural (order-embedding) maps. By condition (d), $y_{i+1} - a_i y_i \in H_i^+ \setminus \{0\}$, hence $f_{i+1}(y_{i+1}) - f_i(y_i) = f_{i+1}(y_{i+1} - a_i y_i) \in G^+ \setminus \{0\}$. This shows that the interval $E = \langle f_i(y_i)\rangle$ is soft and countably generated (see, e.g. [19, Lemma 3.4]).

Let $u = f_1(y_1)$ in G^+, and take this as an order-unit. Denote by s the (unique) state on (G, u); for i in $\mathbb{N}$, let s_i denote the unique state on the simple component $(\mathbb{Z}, H_i^+)$ with respect to the order-unit $u_i = a_{i-1}a_{i-2}\cdots a_1 y_1$. We now check that E is unbounded, that is, $\sup \phi_u(E) = \infty$. By the first part of the proof, $y_{i+1} = p_i c_i + a_i y_i$ where p_i and c_i are chosen in such a way that $p_i c_i > a_i N_{H_i}$. Then, by using condition (b) recurringly, we get

$$\begin{aligned} s(f_{i+1}(y_{i+1})) = s_{i+1}(y_{i+1}) &= \frac{p_i c_i}{a_i a_{i-1}\cdots a_1 y_1} + \frac{a_i y_i}{a_i a_{i-1}\cdots a_1 y_1} > \frac{p_i c_i}{a_i a_{i-1}\cdots a_1 y_1} \\ &> \frac{a_i N_{H_i}}{a_i a_{i-1}\cdots a_1 y_1} = \frac{N_{H_i}}{a_{i-1}\cdots a_1 y_1} > \frac{a_{i-1}^2 N_{H_{i-1}}}{a_{i-1}\cdots a_1 y_1} = \frac{a_{i-1} N_{H_{i-1}}}{a_{i-2}\cdots a_1 y_1} \\ &> \frac{a_{i-1}a_{i-2}^2 N_{H_{i-2}}}{a_{i-2}\cdots a_1 y_1} > \cdots > \frac{a_{i-1}\ldots a_2 a_1 N_{H_1}}{y_1} > a_1^{(i-2)(i-1)}\frac{N_{H_1}}{y_1} , \end{aligned}$$

and so clearly $\sup \phi_u(E) = \infty$.

Now, suppose that $nE = G^+$ for some n. Choose j in $\mathbb{N}$ such that $(N_{H_j} - 1) \geq n$. We have that $f_j((N_{H_j} - 1)x_j) <_G (N_{H_j} - 1)f_i(y_i)$ for all (suitably) large i. This will happen in particular for some $i > j$, which translates into $f_i((N_{H_j} - 1)a_{i-1}a_{i-2}\cdots a_j x_j) <_G f_i((N_{H_j} - 1)y_i)$. Since f_i is an order-embedding, we get $(N_{H_j} - 1)a_{i-1}a_{i-2}\cdots a_j x_j <_{H_i} (N_{H_j} - 1)y_i$ for some $i > j$, in contradiction with condition (e).

Hence, we have constructed a simple group of rank one (G, G^+), containing an interval $E \subset G^+$ such that $\varphi(E) = \infty$ and $nE \neq G^+$ for every n in $\mathbb{N}$. Notice that $A' = (a_i)_{i\geq 1}$ is an infinite subsequence of A, so that $\mathfrak{n} = \mathfrak{n}(A')$ is a generalized integer dividing $\mathfrak{n}(A)$. Moreover, by construction, $G \cong \mathbb{Z}_{\mathfrak{n}}$ (as abelian groups). Thus, for any generalized integer $\mathfrak{m}$ coprime with $\mathfrak{n}$, there exists by Theorem 1.2 a simple Riesz group of rank one $(G(\mathfrak{m}), G^+(\mathfrak{m}))$ such that $G(\mathfrak{m}) \cong \mathbb{Z}_{\mathfrak{n}\mathfrak{m}}$, and an order-embedding $\tau \colon G \to G(\mathfrak{m})$. Then, by condition (iii) in Lemma 3.1, the interval $D = E_\tau = \langle(\tau f_i)(y_i)\rangle$ satisfies that $nD \neq G^+(\mathfrak{m})$ for every n in $\mathbb{N}$. Let

u be an order-unit in G. Since both $S(G,u)$ and $S(G(\mathfrak{m}),\tau(u))$ are singletons with (unique) states s and s' respectively, the affine continuous map $S(\tau)\colon S(G,u)\to S(G(\mathfrak{m}),\tau(u))$ is an homeomorphism with $S(\tau)(s')=s'\tau=s$. Hence, $\sup s'((\tau f_i)(y_i))=\sup(s'\tau)(f_i(y_i))=\sup s(f_i(y_i))=\infty$, whence D is also unbounded. This completes the proof. $\square$

6. The monster example

In this section, we will use the constructions carried out in Theorems 3.4 and 5.2 in order to construct an example of a simple Riesz group of rank one containing unbounded intervals that (simultaneously) enjoy the properties exhibited in those Theorems.

Theorem 6.1. *Let $L=(q_i)_{i\geq 1}$ be a sequence of non-negative, relatively prime integers. Let $I=(a_j)_{j\geq 1}$ be a sequence such that every $a_j\in L$, while each q_i in L appears infinitely many times in I. Let $J=(b_i)_{i\geq 1}$ be a strictly increasing sequence of non-negative integers such that $\gcd(q_i,b_j)=1$ for all $i,j\geq 1$. Let $\mathfrak{n}(I)$ and $\mathfrak{n}(J)$ be the generalized integers associated to I and J respectively. Then, for any generalized integer $\mathfrak{m}$ coprime with $\mathfrak{n}(I)\cdot\mathfrak{n}(J)$, there exists a simple Riesz group of rank one $G(\mathfrak{m})$ such that:*

(i) *For every q_i in L, there is a countably generated interval D_i satisfying $tD_i\neq G(\mathfrak{m})^+$ for $t\leq q_i-1$ and $q_iD_i=G(\mathfrak{m})^+$.*
(ii) *There is an interval $D\subset G(\mathfrak{m})^+$ such that $nD\neq G(\mathfrak{m})^+$ for all n in $\mathbb{N}$.*
(iii) *For some generalized integer $\mathfrak{n}$ dividing $\mathfrak{n}(J)$, the group $G(\mathfrak{m})$ is isomorphic to $\mathbb{Z}_{\mathfrak{n}(I)\cdot\mathfrak{n}\cdot\mathfrak{m}}$ (as abelian groups).*

Proof. We first use the argument in the proof of Theorem 3.4 with the sequence I. In this way we get a simple group of rank one (H,H^+) such that: (a) $H\cong\mathbb{Z}_{\mathfrak{n}(I)}$; (b) For every q_i in L, there is a countably generated interval E_i satisfying $tE_i\neq H^+$ for $t\leq q_i-1$ and $q_iE_i=H^+$. By Proposition 1.1, $(H,H^+)=\varinjlim((\mathbb{Z},H_i^+),l_i\cdot)$, where $(\mathbb{Z},H_i^+)$ is a simple component and $l_i\cdot:(\mathbb{Z},H_i^+)\to(\mathbb{Z},H_{i+1}^+)$ is an order-embedding for all $i\geq 1$. Notice that $\prod_{i\geq 1}l_i=\mathfrak{n}(I)=\prod_{k\geq 1}q_k^{\infty}$. Thus, for each $i\geq 1$, $l_i=\prod_{j=1}^{r_i}q_{k_j}^{n_j}$ for some r_i and n_j in $\mathbb{N}$. Therefore $\gcd(l_i,b_j)=1$ for all $i,j\geq 1$.

Now, fix $(\mathbb{Z},K_1^+)=(\mathbb{Z},H_1^+)$, and apply the argument in the proof of Theorem 5.2, using the sequence J. Thus, we get an inductive system $((\mathbb{Z},K_i^+),a_i\cdot)$, where $(\mathbb{Z},K_i^+)$ is a simple component, $a_i\in J$ and $a_i\cdot:(\mathbb{Z},K_i^+)\to(\mathbb{Z},K_{i+1}^+)$ is an order-embedding for all $i\geq 1$. Moreover, the group $(K,K^+)=\varinjlim((\mathbb{Z},K_i^+),a_i\cdot)$ is a simple group of rank one such that: (a) $K\cong\mathbb{Z}_{\mathfrak{n}}$ for the generalized integer $\mathfrak{n}=\prod_{n\geq 1}a_n$, which divides $\mathfrak{n}(J)$; (b) There is a countably generated interval E such that $nE\neq K^+$ for every $n\geq 1$.

We next define submonoids $G_{i,j}^+$ of the non-negative integers by recurrence on $i,j\geq 0$, as follows:

(a) $G_{0,0}^+=H_1^+=K_1^+$.
(b) For every $i\geq 1$, $G_{i,0}^+=K_{i+1}^+$.
(c) For every $j\geq 1$, $G_{0,j}^+=H_{j+1}^+$.
(d) For every $i,j\geq 1$, $G_{i,j}^+=a_iG_{i-1,j}^++l_jG_{i,j-1}^+$.

By [17, Lemma 2.3(1)], we have that $(\mathbb{Z},G_{i,j}^+)$ is a simple component for every $i,j\geq 0$, and in the following diagram:

$$
\begin{tikzcd}
(\mathbb{Z}, G^+_{0,0}) \arrow[r, "l_1\cdot"] \arrow[d, "a_1\cdot"'] & (\mathbb{Z}, G^+_{0,1}) \arrow[r, "l_2\cdot"] \arrow[d, "a_1\cdot"] & (\mathbb{Z}, G^+_{0,2}) \arrow[r, "l_3\cdot"] \arrow[d, "a_1\cdot"] & \cdots \\
(\mathbb{Z}, G^+_{1,0}) \arrow[r, "l_1\cdot"] \arrow[d, "a_2\cdot"'] & (\mathbb{Z}, G^+_{1,1}) \arrow[r, "l_2\cdot"] \arrow[d, "a_2\cdot"] & (\mathbb{Z}, G^+_{1,2}) \arrow[r, "l_3\cdot"] \arrow[d, "a_2\cdot"] & \cdots \\
(\mathbb{Z}, G^+_{2,0}) \arrow[r, "l_1\cdot"] \arrow[d, "a_3\cdot"'] & (\mathbb{Z}, G^+_{2,1}) \arrow[r, "l_2\cdot"] \arrow[d, "a_3\cdot"] & (\mathbb{Z}, G^+_{2,2}) \arrow[r, "l_3\cdot"] \arrow[d, "a_3\cdot"] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
\tag{6.1}
$$

all squares are commutative and all the maps are order-embeddings (see Proposition 3.3 and [17, Proposition 2.10]).

Let $(G, G^+) = \varinjlim((\mathbb{Z}, G^+_{i,i}), a_i l_i\cdot)$. Then (G, G^+) is a simple group of rank one, and $G \cong \mathbb{Z}_{\mathfrak{n}\cdot\mathfrak{n}(I)}$ by construction. An argument analogous to that in condition (iii) of Proposition 3.3 guarantees the existence of order-embeddings $\sigma\colon (H, H^+) \to (G, G^+)$ and $\tau\colon (K, K^+) \to (G, G^+)$. Thus, for any generalized integer $\mathfrak{m}$ coprime with $\mathfrak{n}(I)\cdot\mathfrak{n}$, there exist by Theorem 1.2 a simple Riesz group of rank one $(G(\mathfrak{m}), G(\mathfrak{m})^+)$ and an order-embedding $\beta\colon (G, G^+) \to (G(\mathfrak{m}), G(\mathfrak{m})^+)$. Clearly, the maps $(\beta\sigma)\colon (H, H^+) \to (G(\mathfrak{m}), G(\mathfrak{m})^+)$ and $(\beta\tau)\colon (K, K^+) \to (G(\mathfrak{m}), G(\mathfrak{m})^+)$ are order-embeddings. Thus, by condition (iii) in Lemma 3.1, the intervals $D = E_{(\beta\tau)}$ and $D_i = (E_i)_{(\beta\sigma)}$ in $(G(\mathfrak{m}), G(\mathfrak{m})^+)$ enjoy the desired properties. $\square$

The example in Theorem 6.1 above allows us to construct a (stably finite) monoid of intervals $W^D_\sigma(G^+)$ over a simple Riesz group G, where D is an unbounded interval such that the representation map φ defined in (3.3) is not injective, even in the case when D is not functionally complete (see [14, Remark 3.4(2)]). Other consequences will be outlined in Section 7.

7. Final comments and remarks

In this section we explore the possible applications of the results obtained in previous sections to the context of K-Theory of multiplier algebras of simple C^*-algebras with real rank zero.

We remind the reader that C^*-algebras are precisely the norm-closed *-subalgebras of $\mathbb{B}(\mathcal{H})$, the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Recall that a (unital) C^*-algebra A has *real rank zero* provided that the set of invertible self-adjoint elements of A is dense in the set of self-adjoint elements of A (see [4]). In case A does not have a unit, then A has real rank zero if, by definition, the minimal unitization $\widetilde{A}$ has real rank zero. We say that a (unital) C^*-algebra A has *stable rank one* if the set of invertible elements of A is dense (see [20], [12]). As with the real rank zero case, if A does not have a unit, then A has stable rank one if $\widetilde{A}$ has. A simple and separable C^*-algebra is said to be *elementary* if it is isomorphic to the algebra of compact operators on a (separable) Hilbert space. This translates into the requirement that the algebra contains minimal projections. We shall be concerned with non-elementary C^*-algebras.

Problem 7.1. Let (G, G^+) be any of the groups obtained in Theorems 3.4, 5.2 or 6.1. Does there exist a simple, separable, non-unital C^*-algebra A with real rank zero and stable rank one for which the ordered group $(K_0(A), K_0(A)^+)$ is order-isomorphic to (G, G^+) ?

We comment below on the relevance of this question, for the consequences that would result given a positive answer. For this, we need to remind the reader of some basic elements in K-Theory that will be needed in our discussion (see, e.g. [3]). Given a C^*-algebra A, we denote by $M_\infty(A) = \varinjlim M_n(A)$, under the maps $M_n(A) \to M_{n+1}(A)$ defined by $x \mapsto \operatorname{diag}(x, 0)$, that is, $M_\infty(A)$ is the algebra of countably infinite matrices over A with only finitely many non-zero entries.

We denote by $V(A)$ the set of equivalence classes of projections in $M_\infty(A)$ under the Murray-von Neuman equivalence $\sim$. This becomes an abelian monoid with operation $[p] + [q] = [\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}]$. This monoid is naturally endowed with the *algebraic pre-order*, denoted by $\leq$, induced by the previous equivalence; namely $[p] \leq [q]$ if p is equivalent to a subprojection of q.

If the C^*-algebra A is represented faithfully as a *-subalgebra of $\mathbb{B}(\mathcal{H})$, for a separable Hilbert space $\mathcal{H}$, and the action is non-degenerate, then we define the *multiplier algebra* $\mathcal{M}(A)$ of A as the C^*-algebra

$$\mathcal{M}(A) = \{x \in \mathbb{B}(\mathcal{H}) \mid xA \subset A \text{ and } Ax \subset A\}.$$

It is well-known that this construction is equivalent to the one obtained by using double centralizers (see, e.g. [27]), and it is of course only relevant in case A does not have a unit itself, since otherwise $\mathcal{M}(A)$ coincides with A. The multiplier algebra, together with the embedding of A as a two-sided closed ideal, provides the solution to the universal problem of adjoining a unit to the algebra A.

If A is a separable (non-unital) C^*-algebra with real rank zero and P is a projection in $\mathcal{M}(A)$, then by [11, Lemma 1.3] we have that PAP also has real rank zero and an approximate unit consisting of an increasing sequence of projections, say (p_n). If, moreover, p is a projection in A, then $p \lesssim P$ if and only if $p \lesssim p_n$ for some $n \geq 1$. In this situation, we define

$$\begin{aligned} \Theta([P]) &= \{[p] \in V(A) \mid p \text{ is a projection in } PM_\infty(A)P\} \\ &= \{[p] \in V(A) \mid [p] \leq [p_n] \text{ for some } n \text{ in } \mathbb{N}\}. \end{aligned}$$

Then $\Theta([P])$ is a countably generated interval in $V(A)$, which is soft precisely when $P \notin A$. Let $D(A) = \Theta([1_{\mathcal{M}(A)}])$. In the case that A has moreover stable rank one, then the map

$$\Theta : (V(\mathcal{M}(A)), [1_{\mathcal{M}(A)}]) \to W_\sigma^{D(A)}(V(A)) \tag{7.1}$$

is a normalized monoid isomorphism (see [19, Theorem 2.4] and also [11, Theorem 1.10]).

For any separable, non-unital, non-elementary simple C^*-algebra with real rank zero and stable rank one, it is well-known that the group $K_0(A)$ is a countable, non-atomic, simple, Riesz group. Because of the existence of an approximate unit of projections, $K_0(A)$ is naturally isomorphic to the Grothendieck group of the monoid $V(A)$. Since A has stable rank one, $V(A)$ has cancellation and it can be identified with $K_0(A)^+$. Let p be any non-zero projection in A and set $u = [p]$ in $V(A)$. If $d = \sup \phi_u(D(A))$ (see also the notation in Section 3), then by composing the map φ defined in (3.3) with the map defined in (7.1), we get a normalized monoid morphism

$$\Phi\colon (V(\mathcal{M}(A)), [1_{\mathcal{M}(A)}]) \to (V(A) \sqcup W_\sigma^d(S_u), d)\,, \tag{7.2}$$

which is an isomorphism if $V(A)$ is furthermore strictly unperforated, see [19, Theorem 3.8]. We now comment on what kind of examples a positive solution to Problem 7.1 would lead to in connection with the results obtained in previous sections.

7.2. *Let $I = (q_n)$ be an increasing sequence of non-negative and relatively prime integers, and let A be a separable, non-unital, non-elementary C^*-algebra with real rank zero and stable rank one such that $(K_0(A), K_0(A)^+)$ is order-isomorphic to the group constructed in Theorem 3.4 (with respect to the sequence I). Then, there exists a sequence of projections $(E_n)_{n\geq 1} \subset \mathcal{M}(A\otimes\mathbb{K})$ such that:*

(i) $\Phi(E_n) = \Phi(1_{\mathcal{M}(A\otimes\mathbb{K})}) = \infty$ *for all* $n \geq 1$.
(ii) $E_{n+1} \not\lesssim E_n$ *for every* $n \geq 1$ *(i.e. E_{n+1} is not equivalent to a subprojection of E_n).*
(iii) *For each $n \geq 1$, $t \cdot E_n \nsim 1_{\mathcal{M}(A\otimes\mathbb{K})}$ whenever $t < q_n$, and $q_n \cdot E_n \sim 1_{\mathcal{M}(A\otimes\mathbb{K})}$.*

Replace A by its stabilization $A\otimes\mathbb{K}$ (where $\mathbb{K}$ is the C^*-algebra of compact operators on a separable Hilbert space), and note that the K_0 group remains the same. So, to verify the above claim, assume that A is stable.

By Theorem 3.4, for every q_n in I, there is a countably generated unbounded interval $D_n \subset K_0^+(A)$ such that $tD_n \neq K_0^+(A)$ for $t \leq q_n - 1$ and $q_nD_n = K_0^+(A)$. Moreover, by Theorem 3.7, we can choose these intervals in such way that $D_{n+1} \leq D_n$ in the algebraic ordering of the monoid of intervals $W_\sigma^{D(A)}(K_0^+(A))$.

Since, as mentioned, we can identify $V(A)$ with $K_0(A)^+$, we can use the isomorphism (7.1), to get a sequence of projections in $\mathcal{M}(A\otimes\mathbb{K})$ by setting $E_n = \Theta^{-1}(D_n)$. Clearly they satisfy properties (i)-(iii).

Notice that, if A is a C^*-algebra satisfying the hypotheses in 7.2 then, for every n, the C^*-algebra $\mathcal{M}(E_n(A\otimes\mathbb{K})E_n)$ is finite. Otherwise, at the level of monoids, $D_n + Y = D_n$ for a non-zero interval Y, and thus, by simplicity of $K_0(A)$, we would conclude that $D_n = K_0^+(A)$, in contradiction with Theorem 3.4. On the other hand

$$M_{q_n}(\mathcal{M}(E_n(A\otimes\mathbb{K})E_n)) \cong \mathcal{M}(M_{q_n}(E_n(A\otimes\mathbb{K})E_n)) \cong \mathcal{M}(A\otimes\mathbb{K})\,,$$

which implies that $M_{q_n}(\mathcal{M}(E_n(A\otimes\mathbb{K})E_n))$ is not finite. This kind of behaviour has been exhibited in concrete examples constructed by Rørdam (see [21]). There are even simple examples, but they don't have real rank zero (see [22], [23]).

The existence of examples as in 7.2 would provide us with examples of C^*-algebras of real rank zero that fail to have weak cancellation in the sense of Brown and Pedersen (see [5]). They would also give a solution to the Fundamental Separativity Problem (see, e.g. [1]).

7.3. *Let A be a separable, non-unital, non-elementary C^*-algebra with real rank zero and stable rank one such that $(K_0(A), K_0(A)^+)$ is order-isomorphic to the group constructed in Theorem 5.2. There exists then a projection E in $\mathcal{M}(A\otimes\mathbb{K})$ such that:*

(i) $\Phi(E) = \Phi(1_{\mathcal{M}(A\otimes\mathbb{K})}) = \infty$.
(ii) $n \cdot E \nsim 1_{\mathcal{M}(A\otimes\mathbb{K})}$ *for every* $n \geq 1$.

To check this, use Theorem 5.2, to find a countably generated unbounded interval $D \subset K_0^+(A)$ such that $nD \neq K_0^+(A)$ for every $n \geq 1$. Then, using the isomorphism (7.1), we get a projection $E = \Theta^{-1}(D)$ in $\mathcal{M}(A\otimes\mathbb{K})$ satisfying the required properties.

Notice that if A is a C^*-algebra satisfying the hypotheses in 7.3, then we have an answer to an implicit question posed in [14, Remark 3.4(2)]. Namely, if (G, G^+) is a simple Riesz group containing an interval $D \subseteq G^+$ such that $\varphi(D) = \varphi(G^+) = \infty$, but $nD \neq G^+$ for every n in $\mathbb{N}$, then if $Y \in W_\sigma^D(G^+)$ and $nD + Y = nD$, the simplicity of (G, G^+) would imply that $(n+1)D = nD$, and thus $nD = G^+$, contradicting the hypothesis (i.e. $W_\sigma^D(G^+)$ is a stably finite monoid, see e.g. [14]). So, $nD \neq mD$ whenever $n \neq m$, but still $\varphi(nD) = \infty$. Hence, it might be possible to construct a unital, simple C^*-algebra A with real rank zero and stable rank one, such that the multiplier algebra $\mathcal{M}(A \otimes \mathbb{K})$ contains a non-zero projection E with $\mathcal{M}(E(A \otimes \mathbb{K})E)$ stably finite, but with identically infinite scale ([19]). Moreover, according to [21, Proposition 3.6] (also see [18, Theorem 2.10]), $E(A \otimes \mathbb{K})E$ would not be a stable algebra. The existence of such an example would fix the exact limits of application of [18, Proposition 2.11].

7.4. *Let $I = (q_n)$ be an increasing sequence of relatively prime non-negative integers. Let A be a separable, non-unital, non-elementary C^*-algebra with real rank zero and stable rank one such that $(K_0(A), K_0(A)^+)$ is order-isomorphic to the group constructed in Theorem 6.1 (with respect to the sequence I). There exists then a sequence of projections $(E_n)_{n\geq 1}$ and a projection E in $\mathcal{M}(A \otimes \mathbb{K})$ such that:*

(i) $\Phi(E_n) = \Phi(1_{\mathcal{M}(A\otimes\mathbb{K})}) = \infty$ *for every* $n \geq 1$.
(ii) $E_{n+1} \lnsim E_n$ *for every* $n \geq 1$.
(iii) *For every* $n \geq 1$, $t \cdot E_n \nsim 1_{\mathcal{M}(A\otimes\mathbb{K})}$ *whenever* $t < q_n$, *and* $q_n \cdot E_n \sim 1_{\mathcal{M}(A\otimes\mathbb{K})}$.
(iv) $\Phi(E) = \Phi(1_{\mathcal{M}(A\otimes\mathbb{K})}) = \infty$.
(v) $n \cdot E \nsim 1_{\mathcal{M}(A\otimes\mathbb{K})}$ *for every* $n \geq 1$.

Hence, in view of 7.2 and 7.3, the existence of a C^*-algebra A satisfying the hypotheses in 7.4 would imply that the multiplier algebra $\mathcal{M}(A \otimes \mathbb{K})$ contains projections having the special behaviors stated in there.

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